Efficient Data Structure for Representing & Simplifying Simplicial Complexes in High Dimensions

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Motivation

Reconstruction → Simplification → Analysis

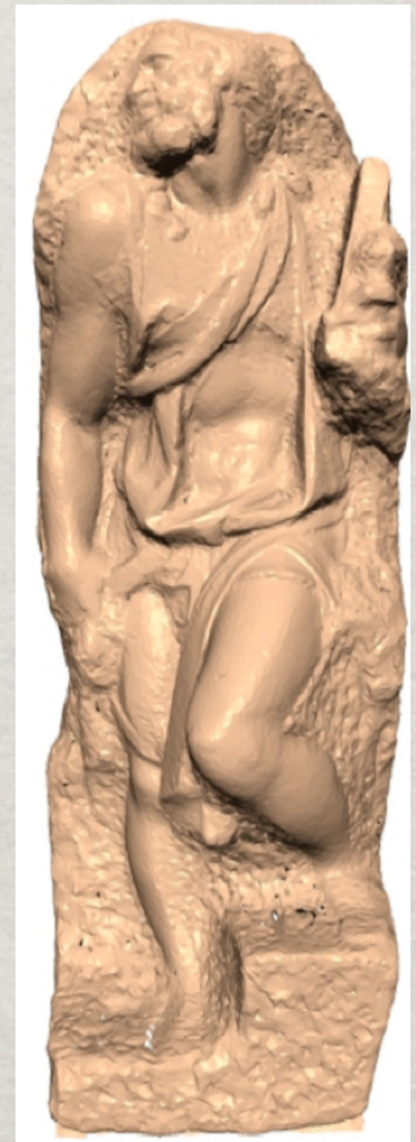
$\simeq$

3D scans

large
data sets

$\approx 400,000,000$ triangles

gigantic
simplicial complex

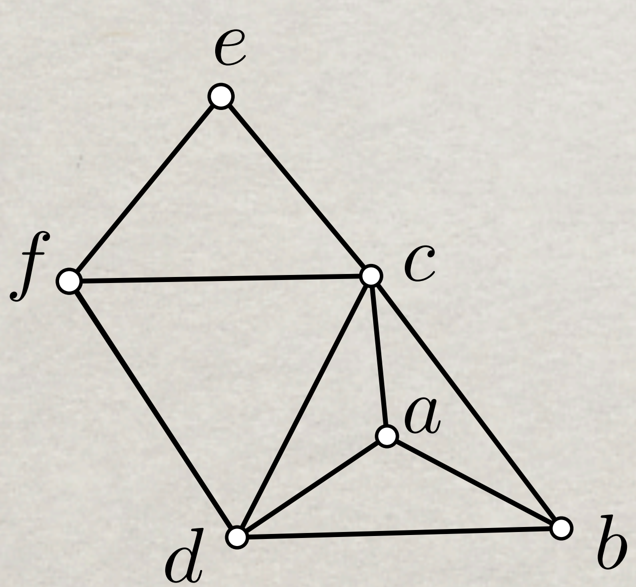


$\approx 100,000$ triangles

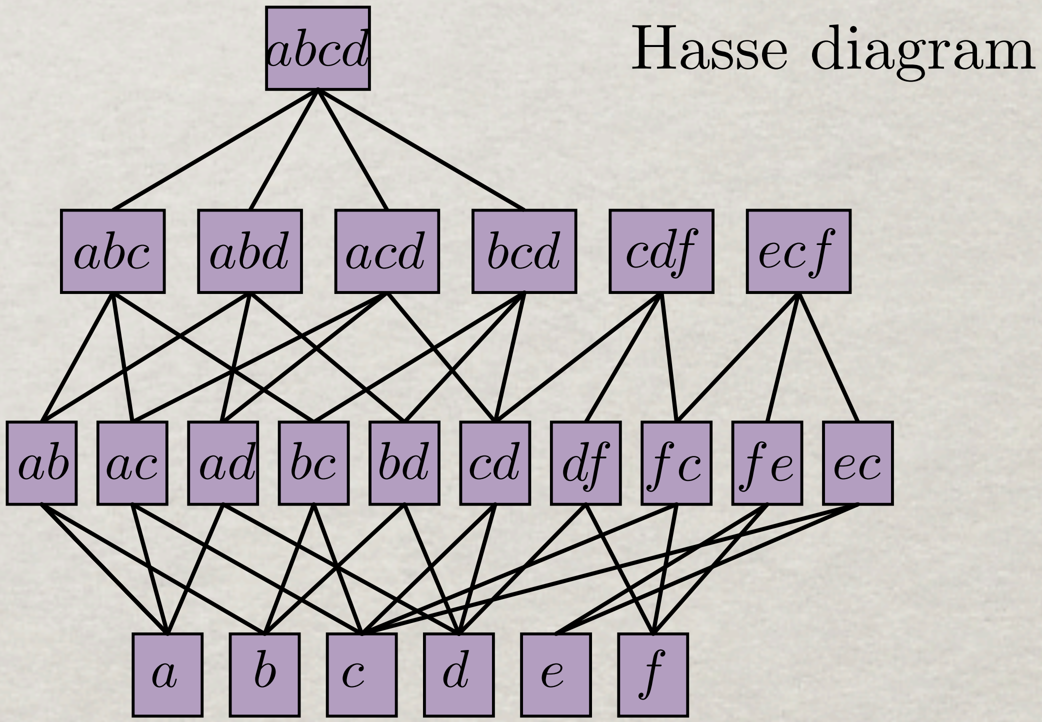
reduced
simplicial complex

Flag Complexes

- Flag G = largest simplicial complex whose 1-skeleton is G.
- $\{v_0, \ldots, v_k\} \in \text{Flag}\, G \iff v_i v_j \in G \;\; \forall i, j$



G



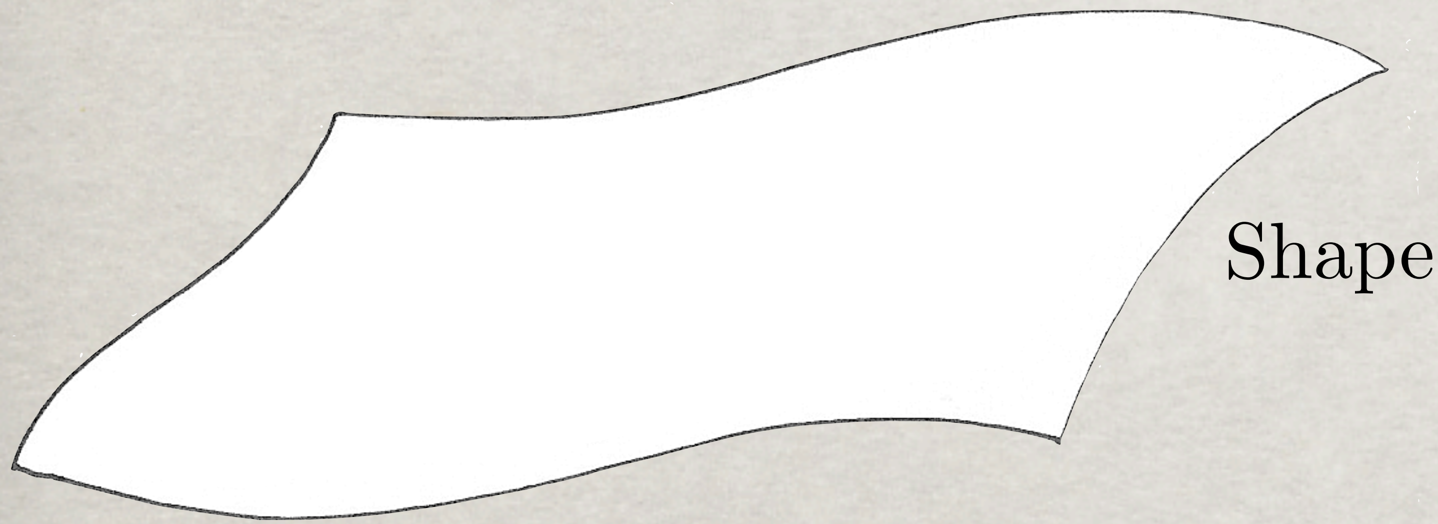
Flag G

Flag complexes have a compressed form of storage

Shape Reconstruction

INPUT

Point cloud P



$G_\alpha(P)$ = proximity graph

$$pq \in G_\alpha(P) \iff d(p,q) \leq 2\alpha$$

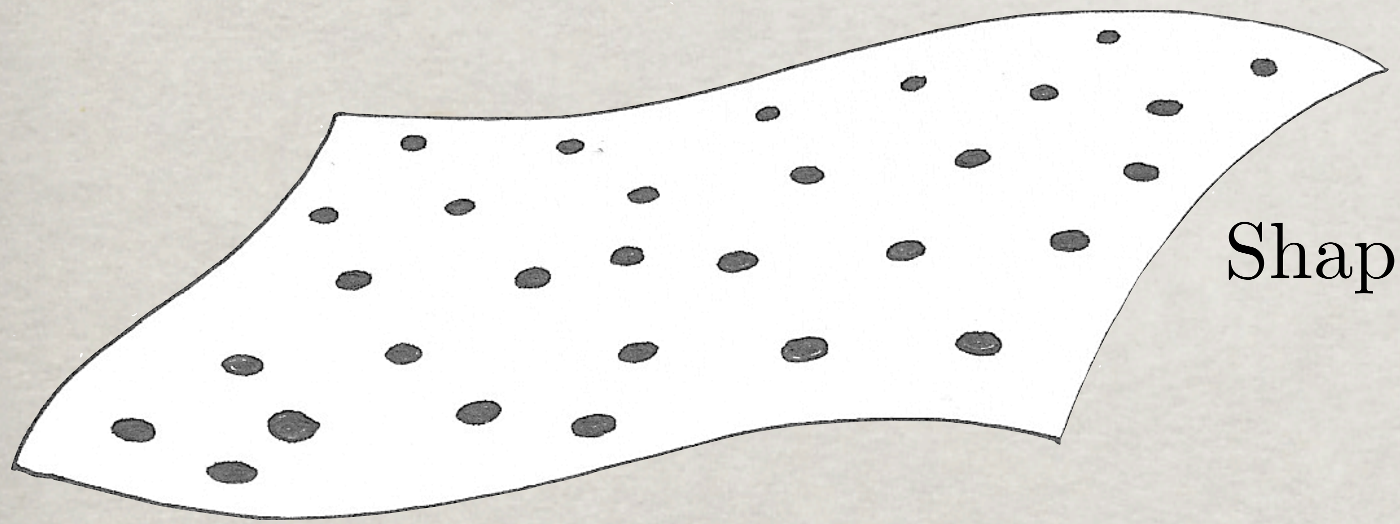
OUTPUT

Flag $G_\alpha(P)$ = Rips complex

Shape Reconstruction

INPUT

Point cloud P



Shape

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Shape Reconstruction

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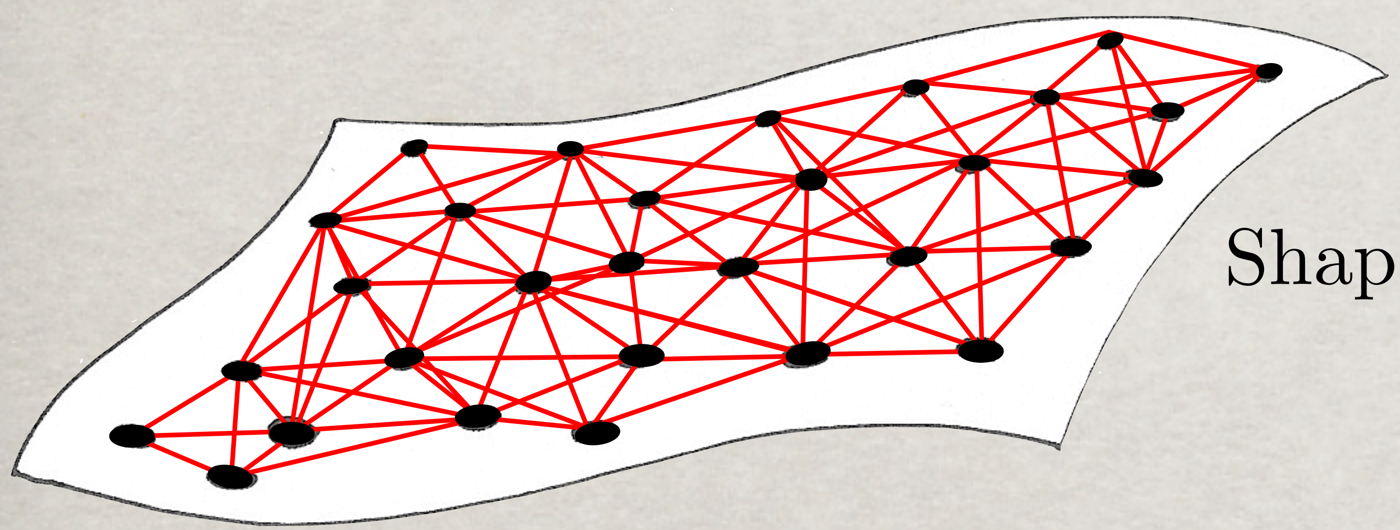

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Shape Reconstruction

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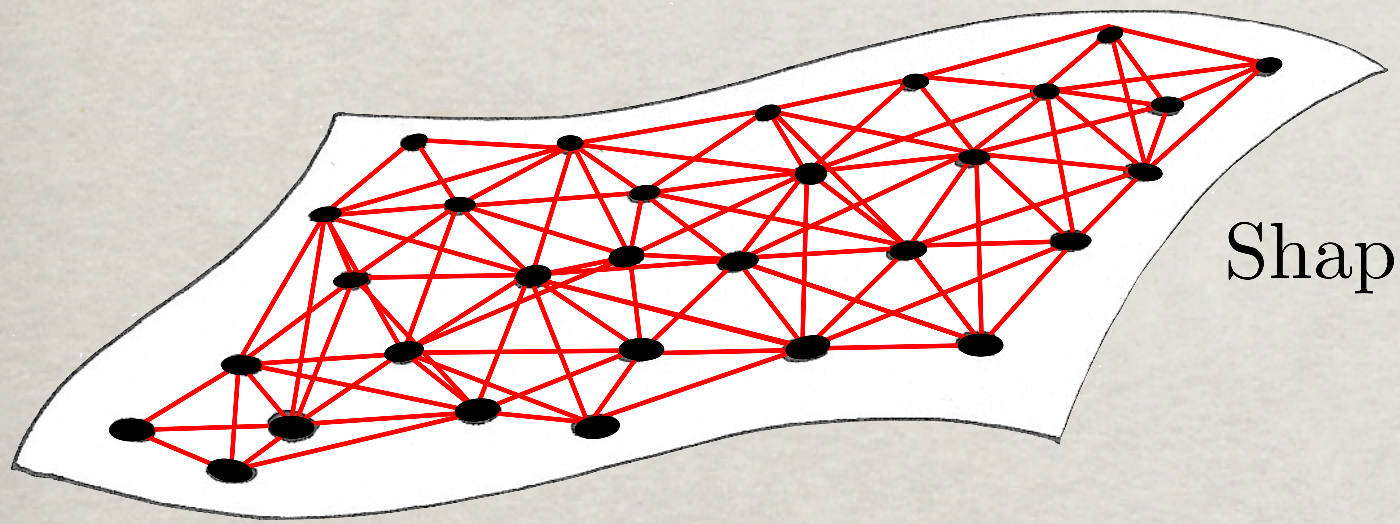


$G_\alpha(P)$ = proximity graph

$pq \in G_\alpha(P) \iff d(p,q) \leq 2\alpha$

OUTPUT

Flag $G_\alpha(P)$ = Rips complex

$\simeq$ Shape

for sampling conditions stated in

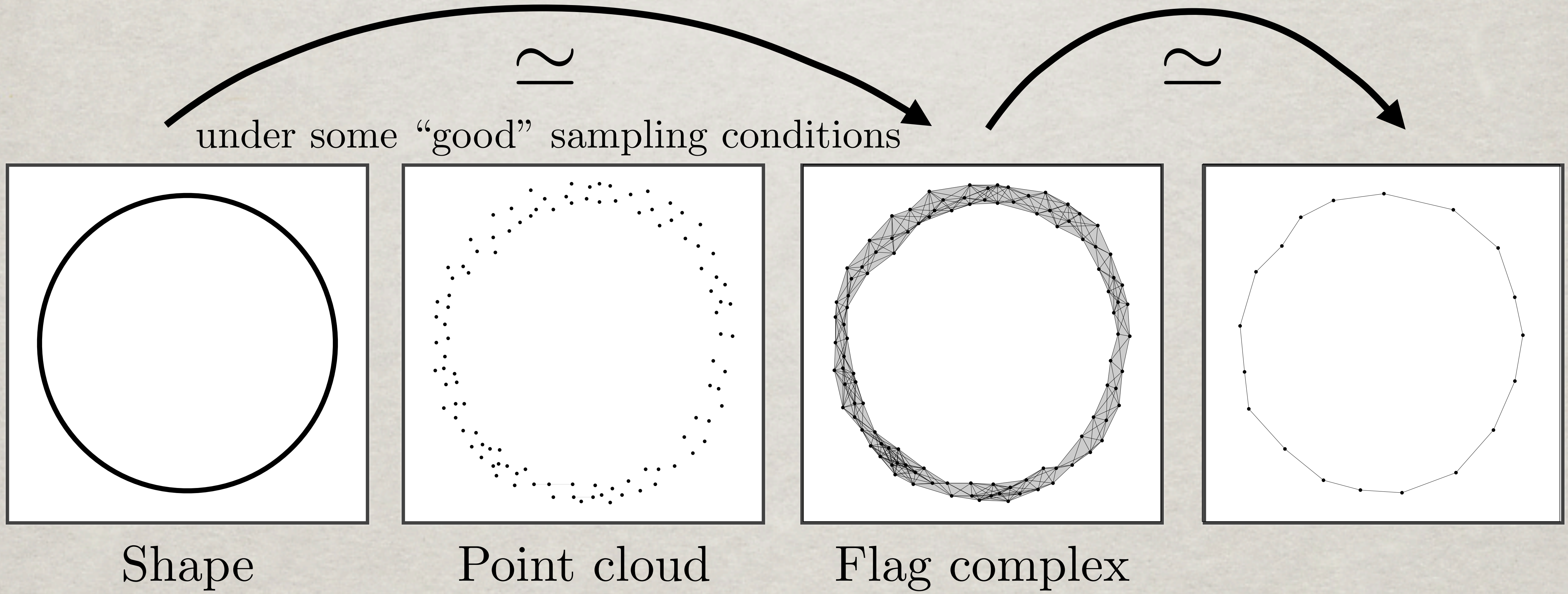
[AL 2010] when $d = d_\infty$

[ALS 2011] when $d = d_2$

Overview

Part I: Guarantees

Part II: Simplification

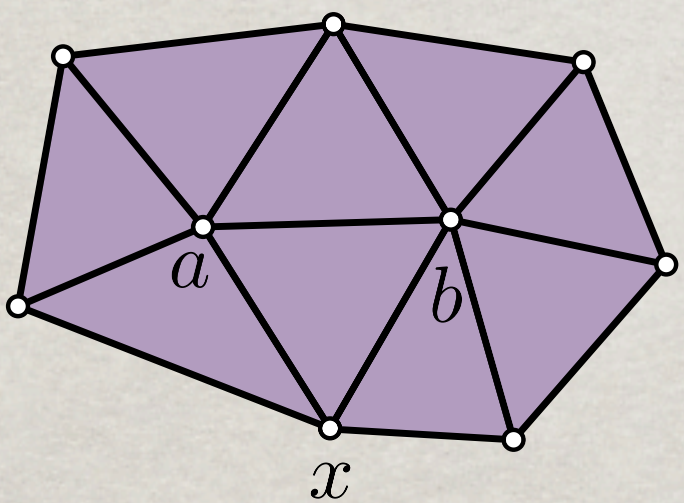


can be high dimensional !

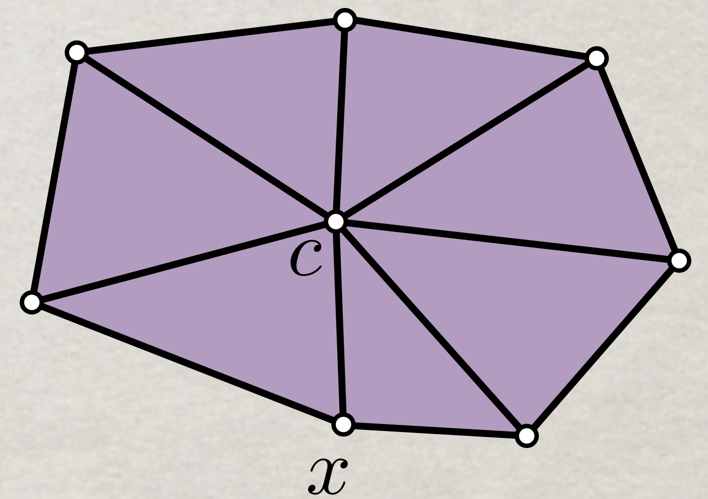
Edge contraction

operation that identifies vertices a and b to vertex c

$$K \xrightarrow{ab \mapsto c} K'$$



$\{\cdots, a, b, x, ab, ax, bx, abx, \cdots\}$



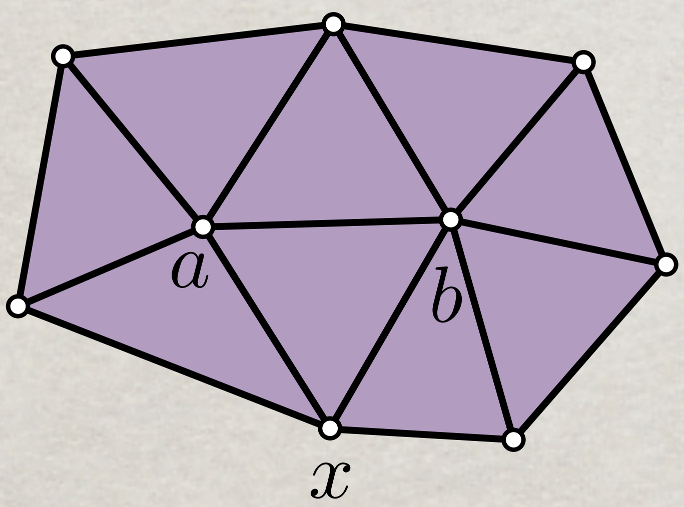
$\{\cdots, c, x, cx, \cdots\}$

Edge contraction

operation that identifies vertices a and b to vertex c

$$K \xrightarrow{ab \mapsto c} K' = \{f(\sigma) \mid \sigma \in K\}$$

$$f(v) = \begin{cases} c & \text{if } v \in \{a, b\} \\ v & \text{if } v \notin \{a, b\} \end{cases}$$



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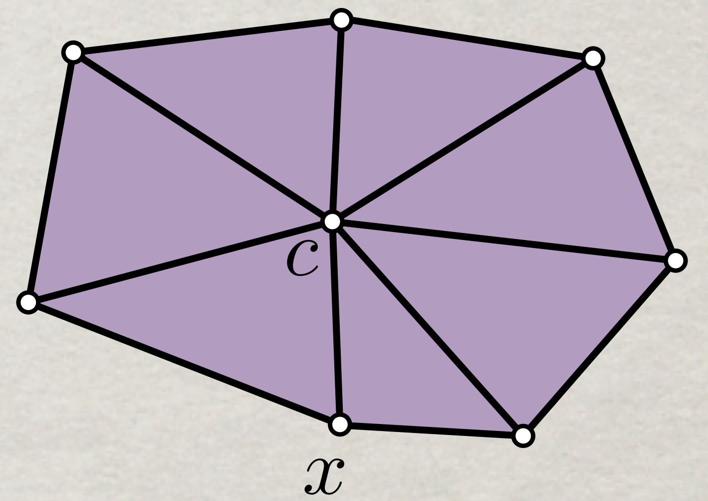

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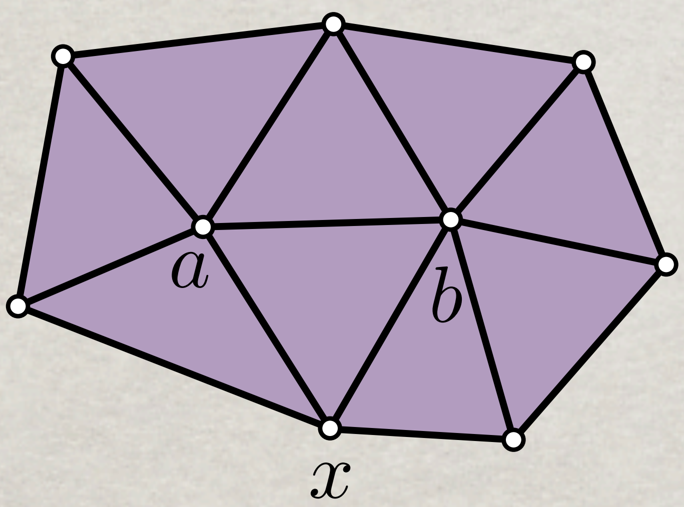
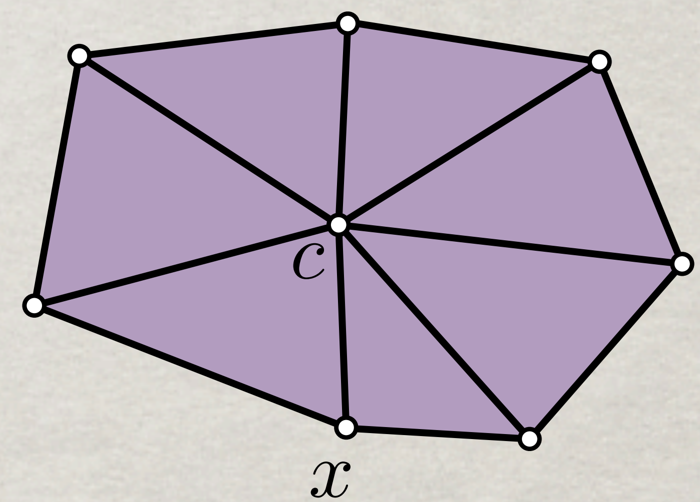


$\{\cdots, a, b, x, ab, ax, bx, abx, \cdots\}$

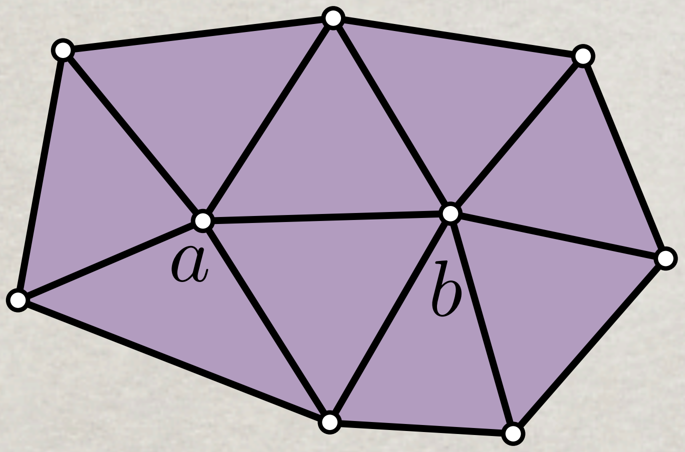


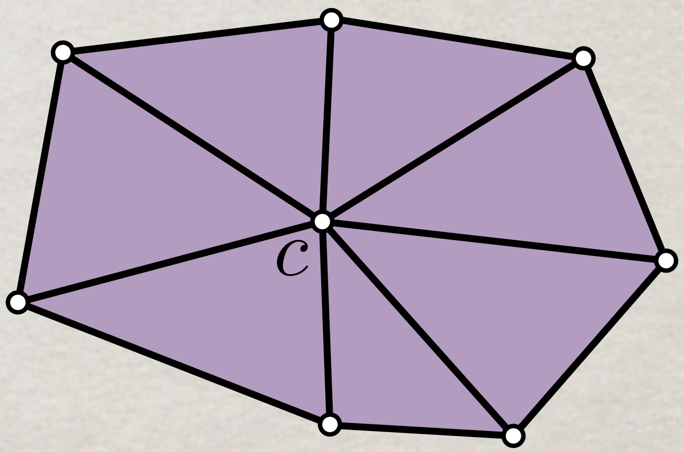
$\{\cdots, c, x, cx, \cdots\}$

- What if the result is not a flag complex?
- How to preserve homotopy type?

Edge contraction

$K = \mathrm{Flag}\, K^{(1)}$ $\xrightarrow{ab \mapsto c}$ $K' = \mathrm{Flag}\, K'^{(1)}$





Edge contraction

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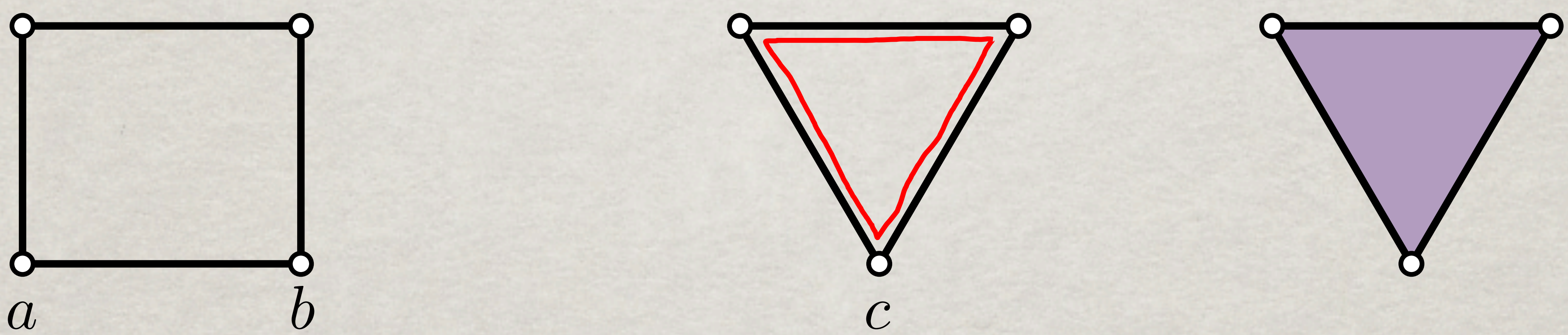

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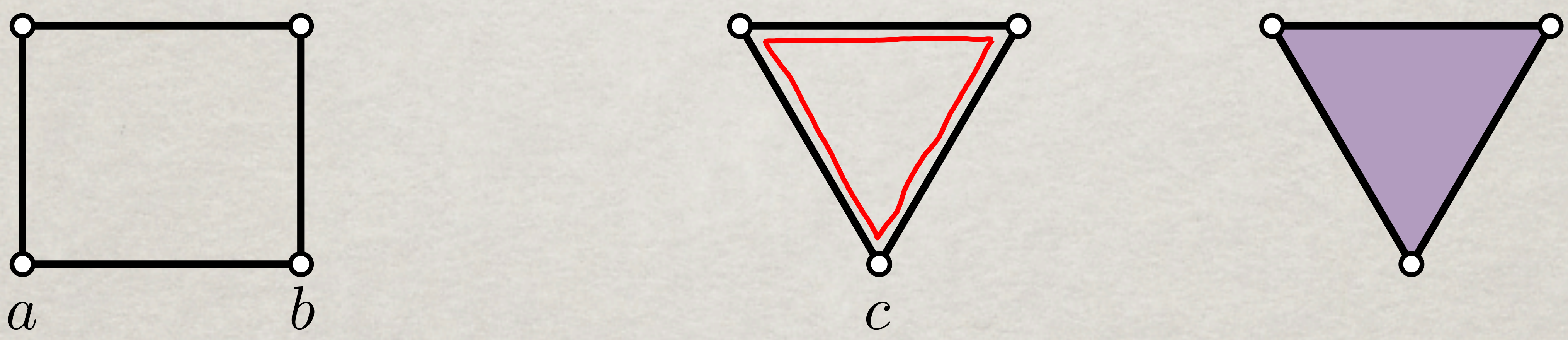
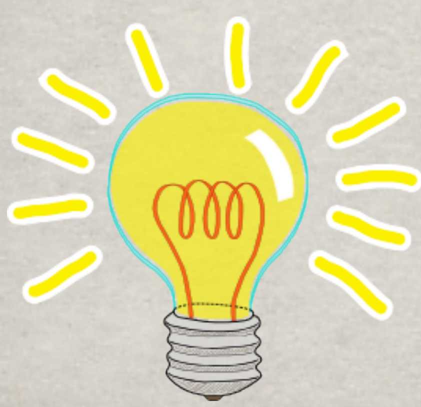




Encode a simplicial complex K by storing the pair:

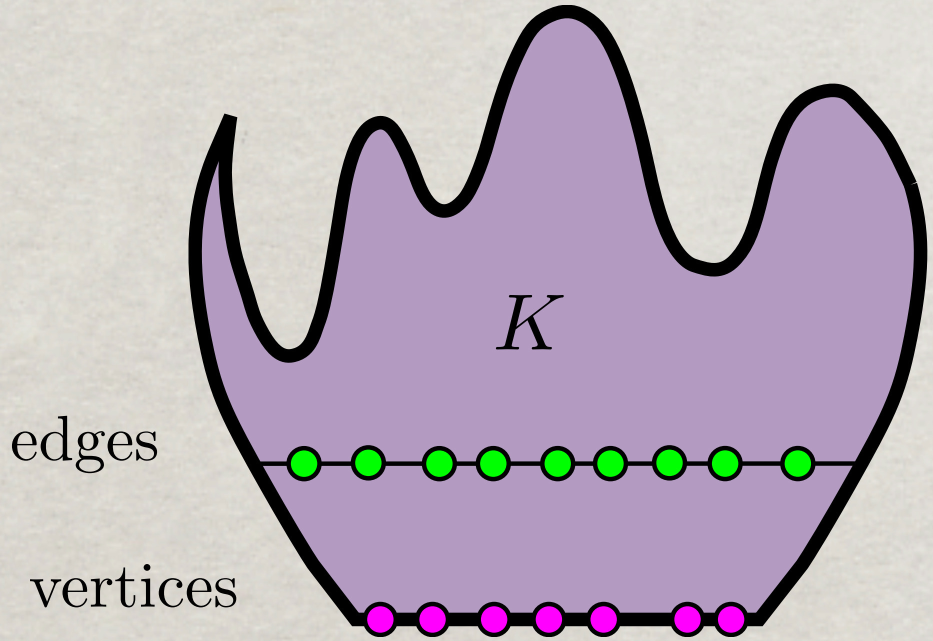
$$(\; K^{(1)}, \mathrm{Blockers}(K) \;)$$

vertices and edges

indicates how much
K differs from $\mathrm{Flag}\, K^{(1)}$

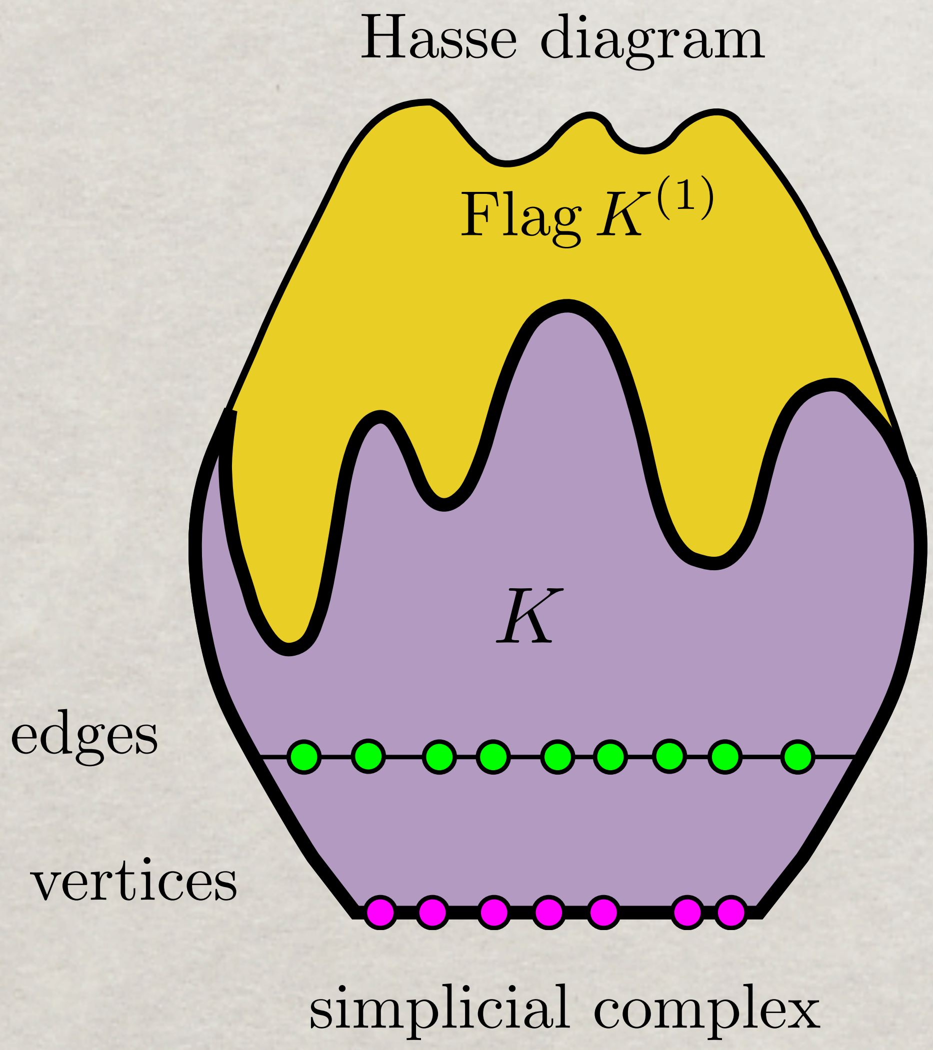
Data Structure for Simplicial complexes

Hasse diagram

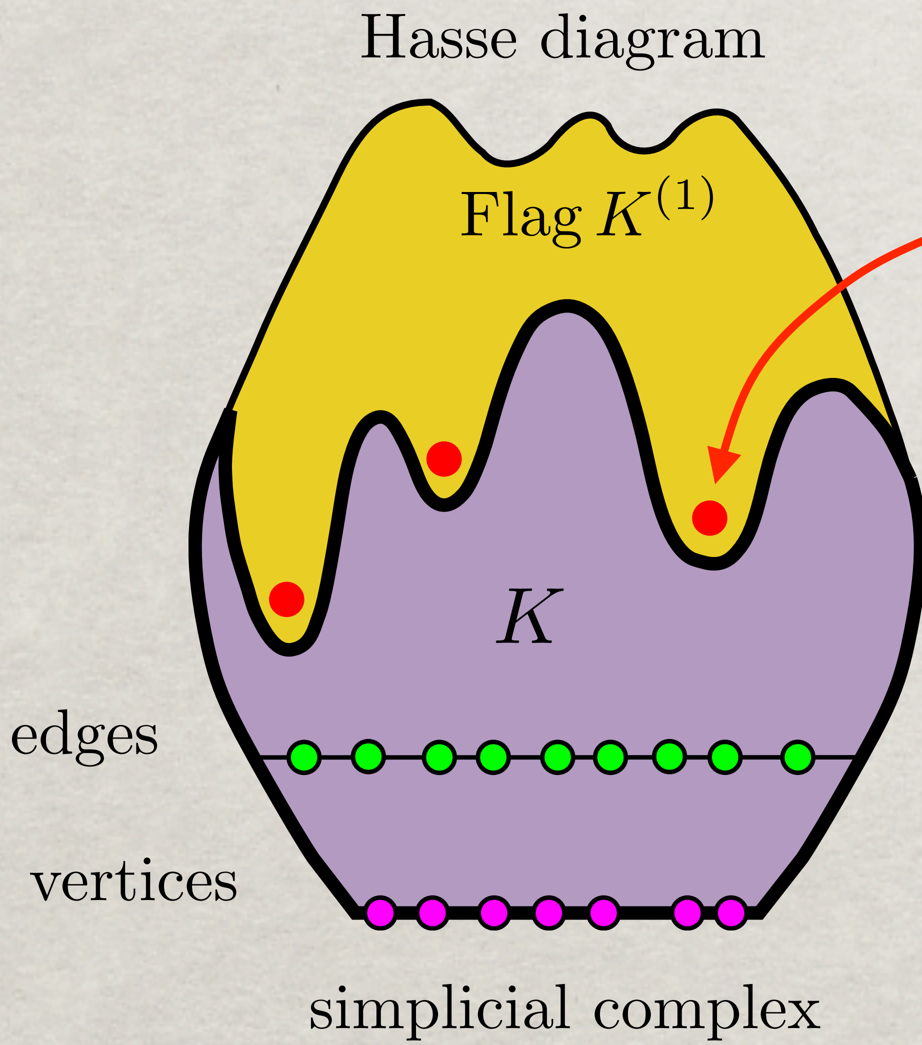


simplicial complex

Data Structure for Simplicial complexes

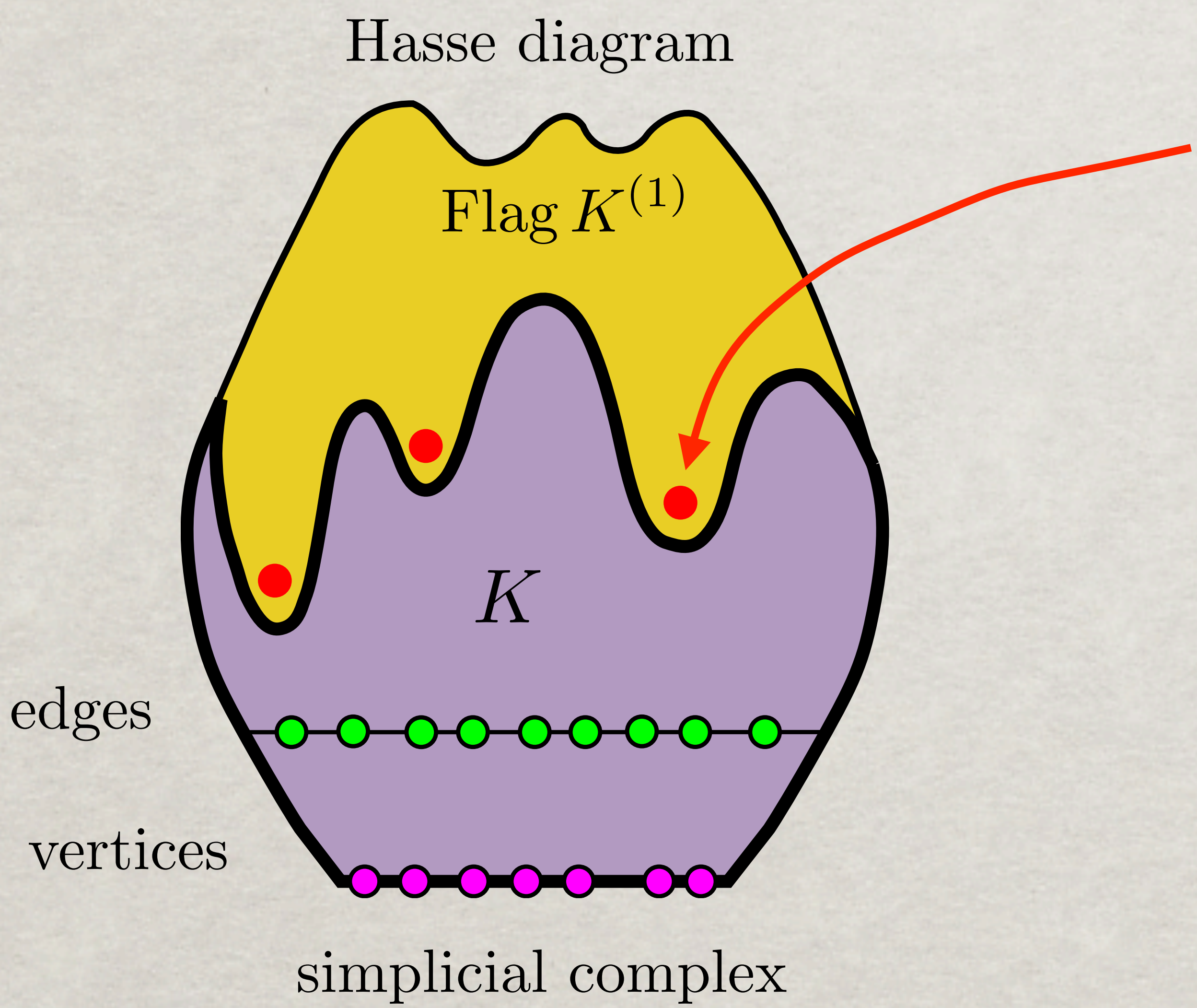


Data Structure for Simplicial complexes



Blockers of K are inclusion-minimal simplices of $\operatorname{Flag} K^{(1)} \setminus K$

Data Structure for Simplicial complexes



Blockers of K are inclusion-minimal simplices of $\text{Flag}\, K^{(1)} \setminus K$

σ blocker of K

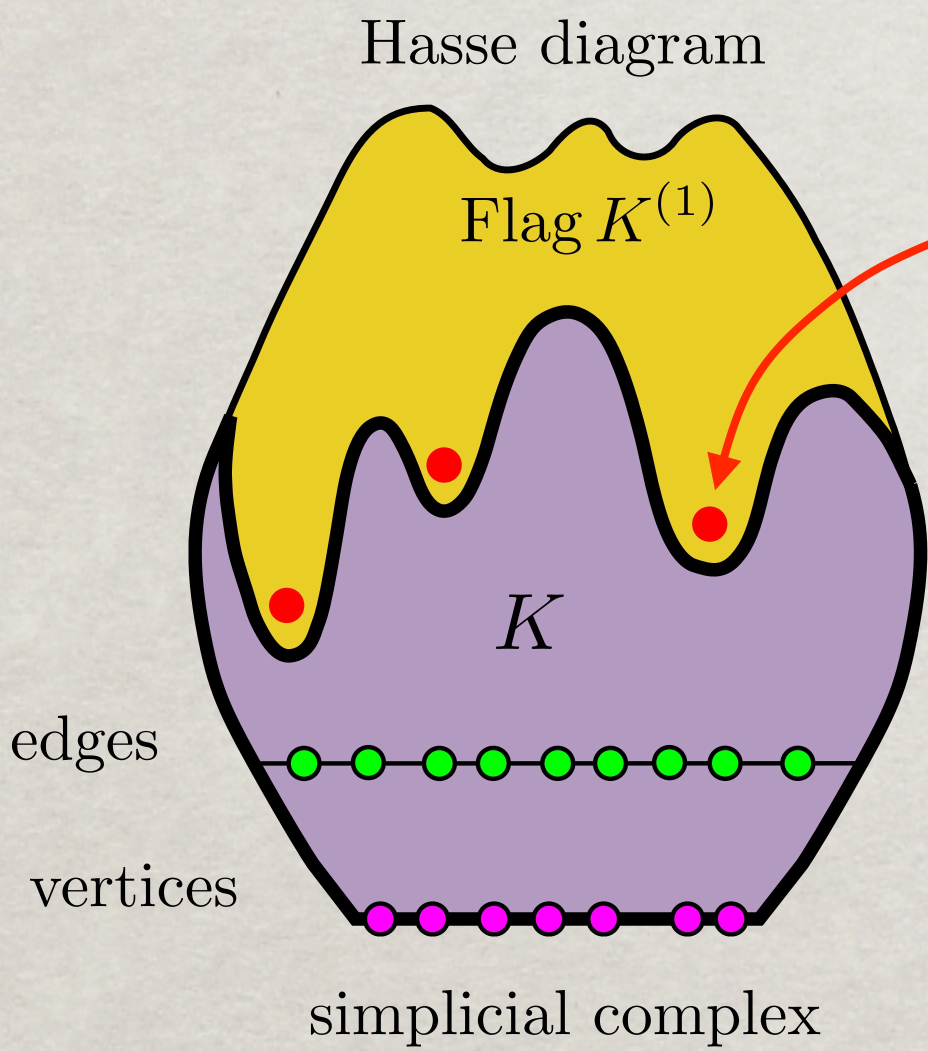
$\Longleftrightarrow$

$\dim \sigma \geq 2$

$\sigma \notin K$

$\forall \tau \subsetneq \sigma, \ \tau \in K$

Data Structure for Simplicial complexes



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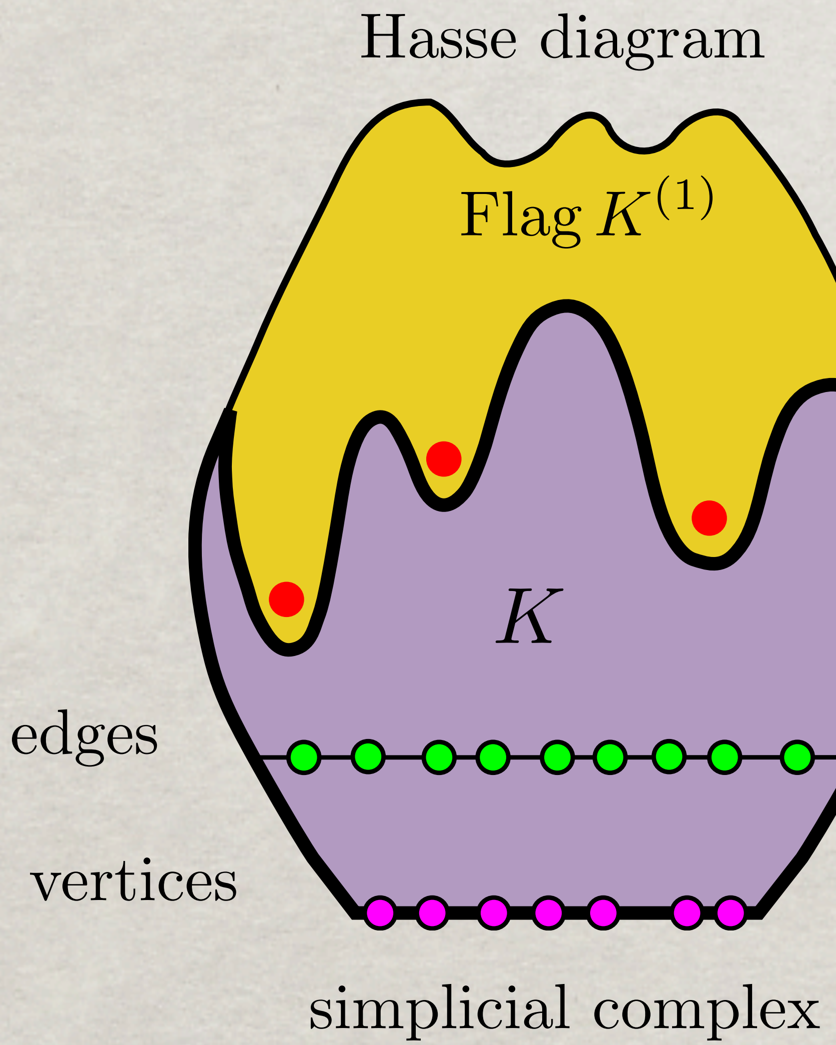
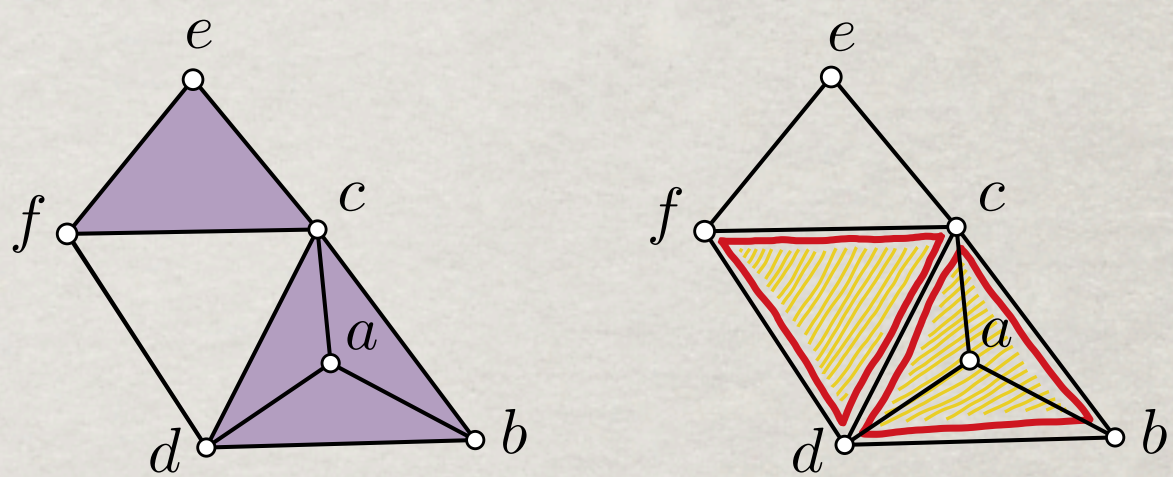
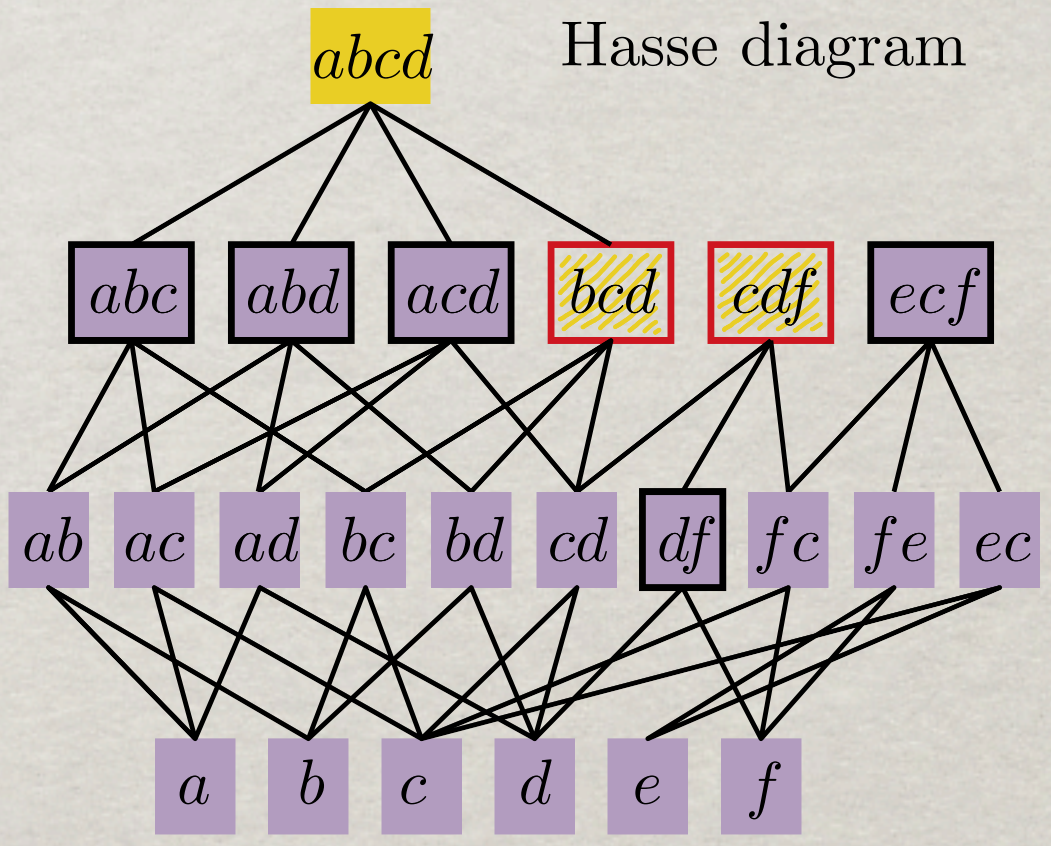
Encode a simplicial complex K by storing the pair:

$(\ K^{(1)}, \text{Blockers}(K)\ )$

A small example

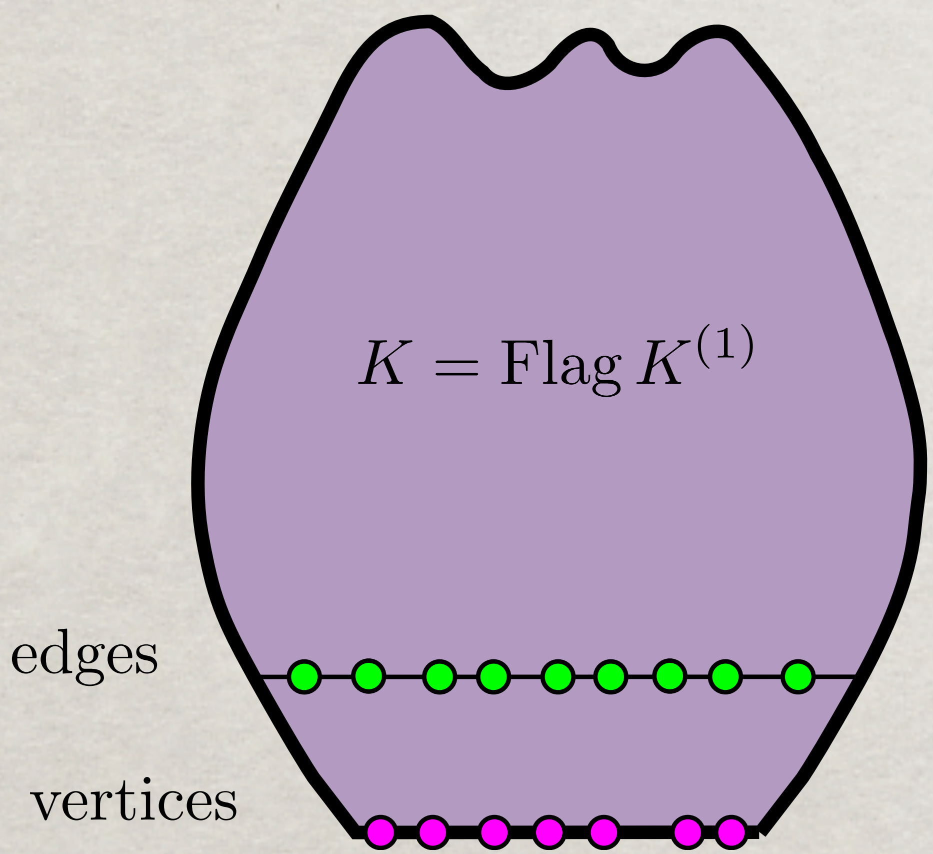






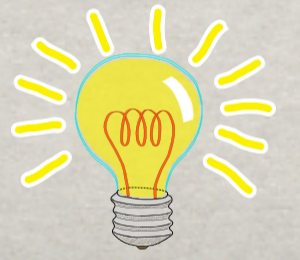
$$\text{Blockers}(K) = \{bcd, cdf\}$$

Data Structure for Simplicial complexes



Blockers of K are inclusion-minimal simplices of $\operatorname{Flag} K^{(1)} \setminus K$

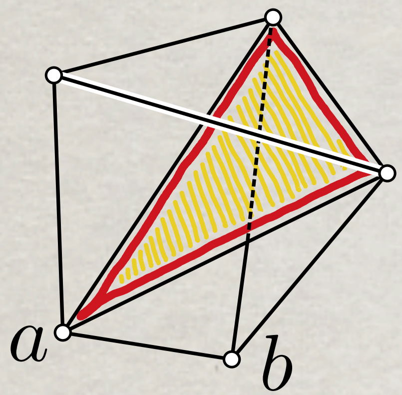
If K is a flag complex
$\text{Blockers}(K) = \emptyset$

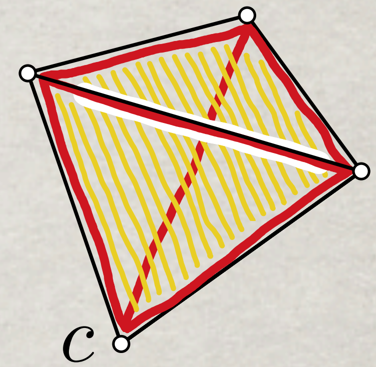


For a flag complex K, the pair reduces to:

$$(K^{(1)} , \emptyset)$$

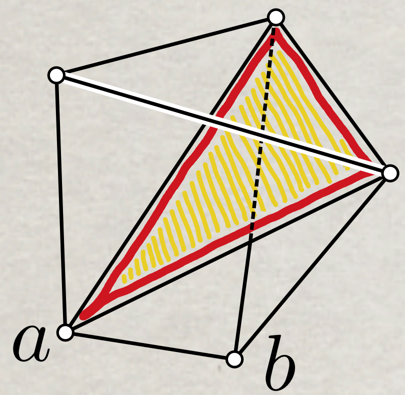
Updating Data Structure

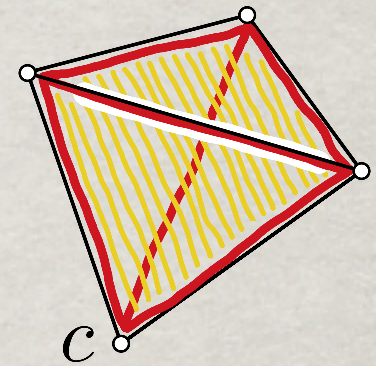




$K = (\, K^{(1)} \,, \text{Blockers}(K)\,)$ $\xrightarrow{ab \mapsto c}$ $K' = (\, K'^{(1)} \,, \text{Blockers}(K')\,)$

Updating Data Structure





$$K = (\,K^{(1)}\,,\ \mathrm{Blockers}(K)\,) \xrightarrow{\ ab \mapsto c\ } K' = (\,K'^{(1)}\,,\ \mathrm{Blockers}(K')\,)$$

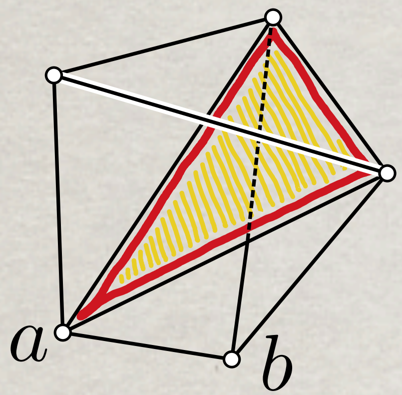
Lemma 1. $c\sigma \in \mathrm{Blockers}\,(K')$ *with* $\sigma \subset \mathrm{Vert}(K) \setminus \{a,b\}$ *and* $\dim \sigma \geq 1$ *iff:*

(i) $\sigma \in K$*; for all* $\tau \subsetneq \sigma$, $\tau \in \mathrm{Lk}(a) \cup \mathrm{Lk}(b)$*;*

(ii) $\sigma = \alpha\beta$ *with* $a\beta \in \mathrm{Blockers}_0(K)$ *and* $b\alpha \in \mathrm{Blockers}_0(K)$*,*

where $\mathrm{Blockers}_0(K) = \mathrm{Blockers}\,(K) \cup$ *complement of* $K^{(1)}$

Updating Data Structure



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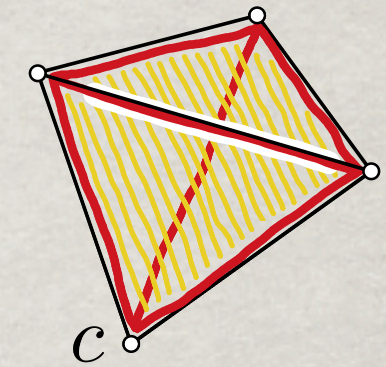

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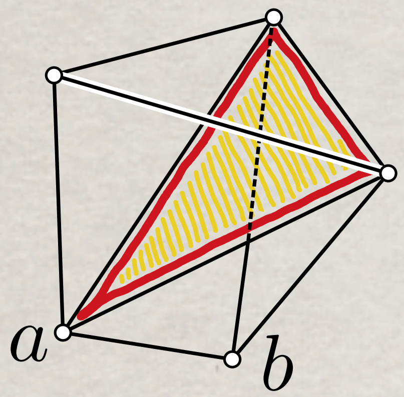
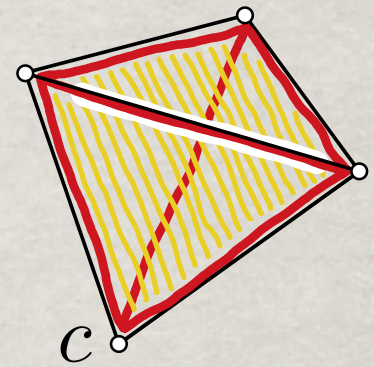
$\mathrm{Lk}\, v = (\,(\mathrm{Lk}\, v)^{(1)}\,,\ \mathrm{Blockers}(\mathrm{Lk}\, v)\,)$

Updating Data Structure





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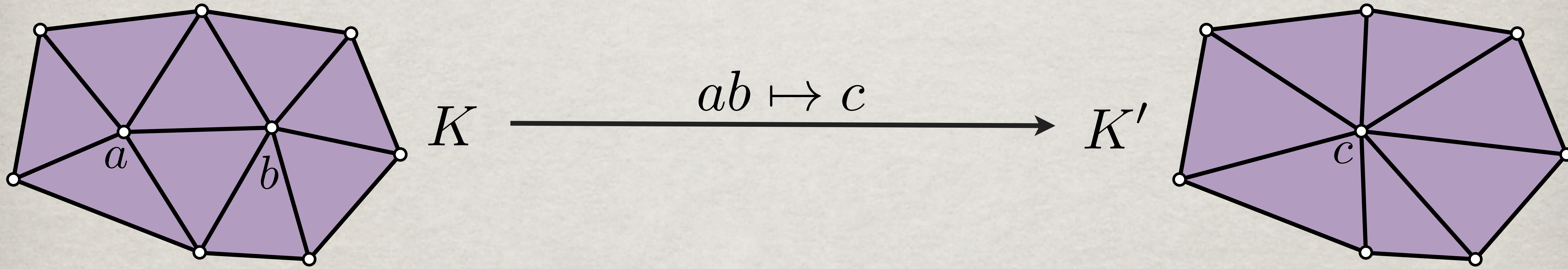
$$\mathrm{Lk}\, v = (\, (\mathrm{Lk}\, v)^{(1)} \,,\, \mathrm{Blockers}(\mathrm{Lk}\, v)\,)$$

If no blockers "around" a and b, costs in $\tilde{O}(\sharp\mathrm{neighbors}(a) \times \sharp\mathrm{neighbors}(b))$

Edge contraction

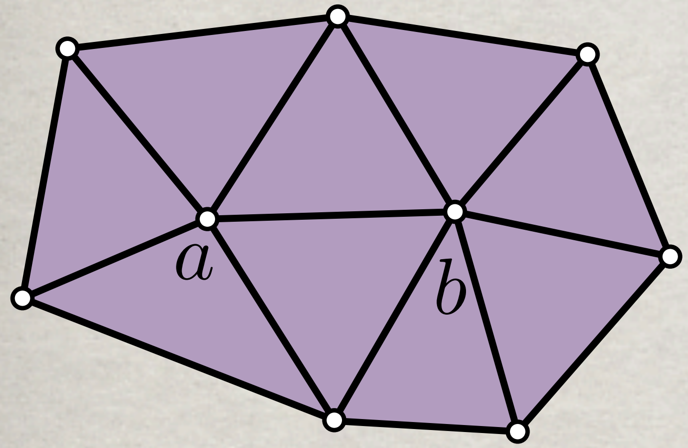
How to preserve homotopy type?



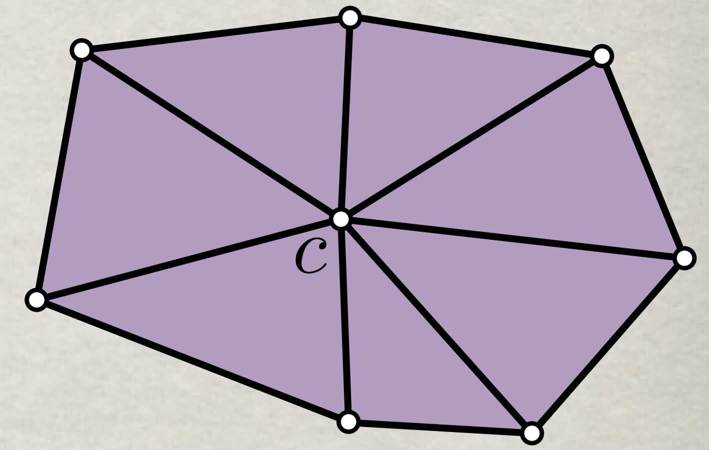
Edge contraction

[Dey, Edelsbrunner, Guha & Nekhayev 1999]

If K triangulates a 2- or 3-manifold



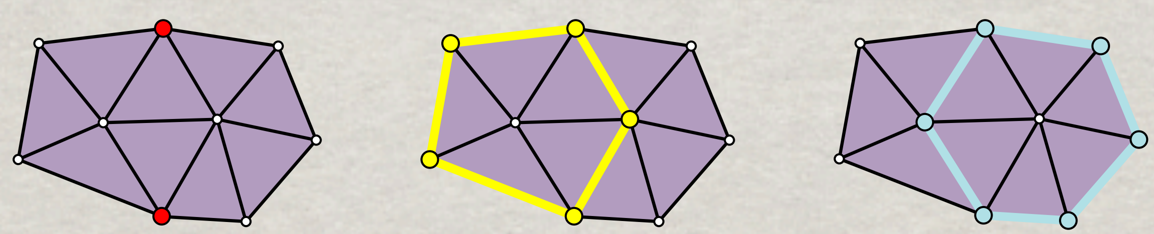
$$K \xrightarrow{ab \mapsto c} K'$$



$\approx$

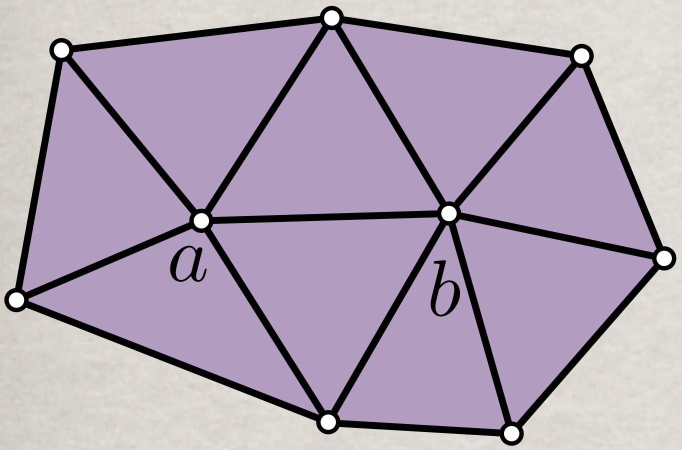
iff

$$\operatorname{Lk} ab = \operatorname{Lk} a \cap \operatorname{Lk} b$$

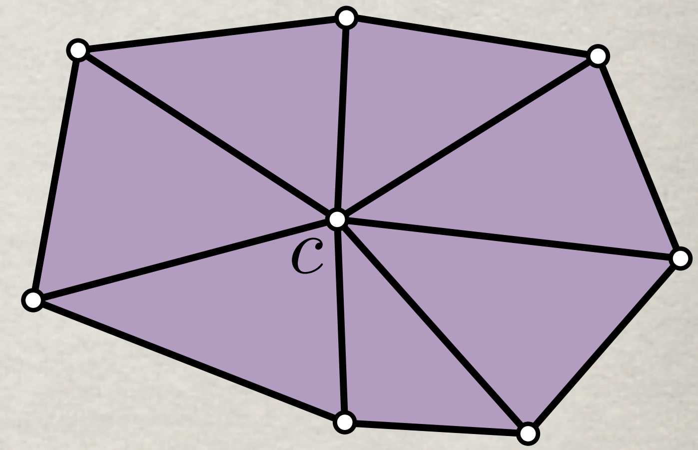


Edge contraction

For arbitrary simplicial complexes, we established that:



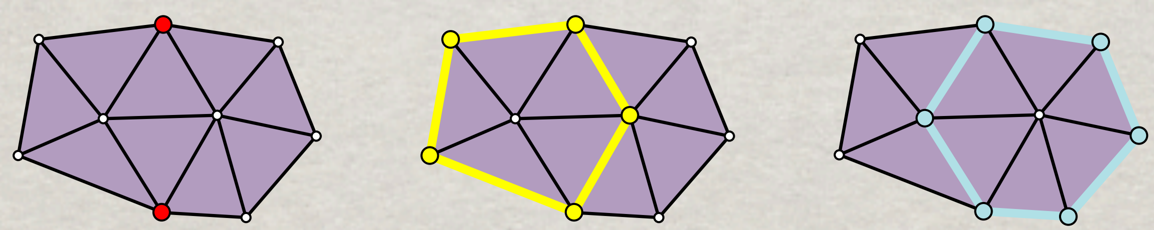
$$K \xrightarrow{\;ab \mapsto c\;} K'$$



$\simeq$

whenever

$$\operatorname{Lk} ab = \operatorname{Lk} a \cap \operatorname{Lk} b$$



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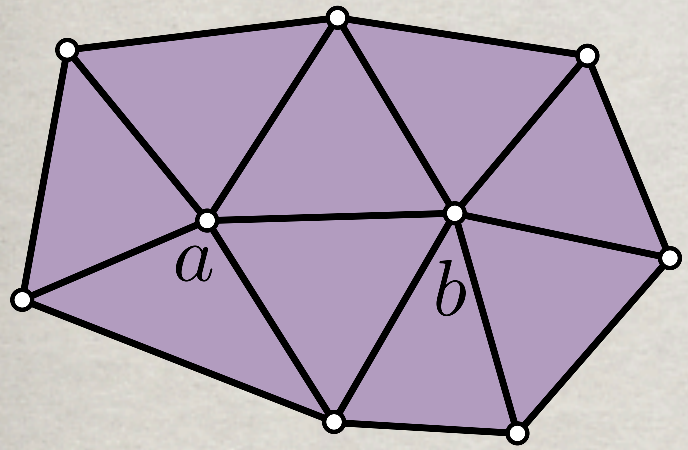
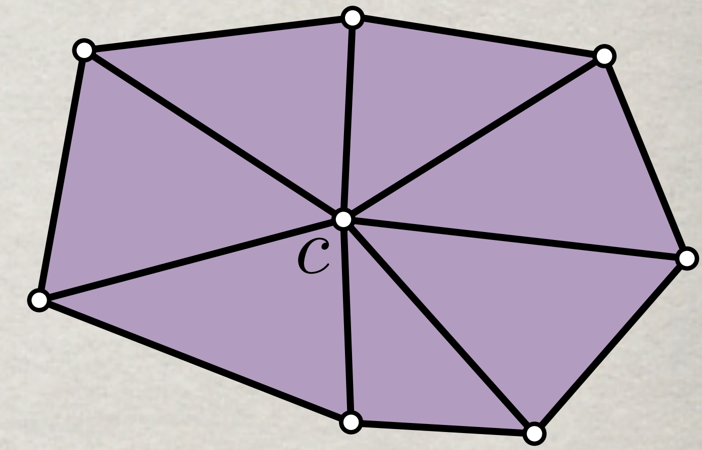


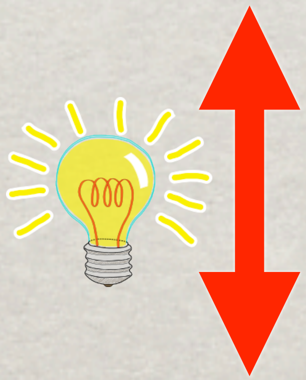
$$K \xrightarrow{\;ab \mapsto c\;} K'$$



$\simeq$

whenever

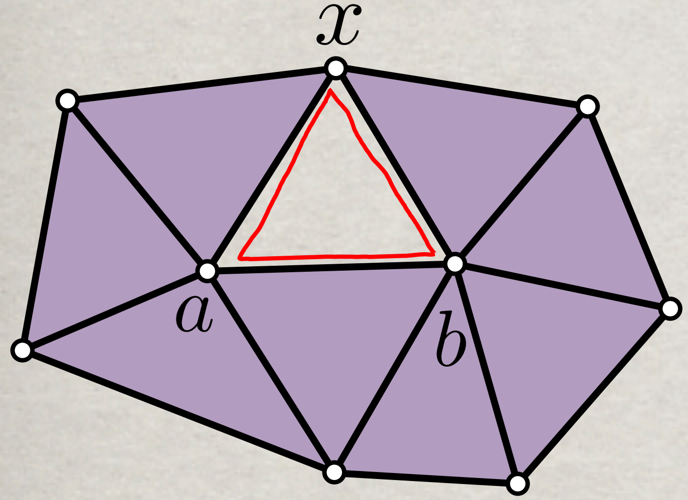
$$\mathrm{Lk}\, ab = \mathrm{Lk}\, a \cap \mathrm{Lk}\, b$$



No blocker of K contains ab

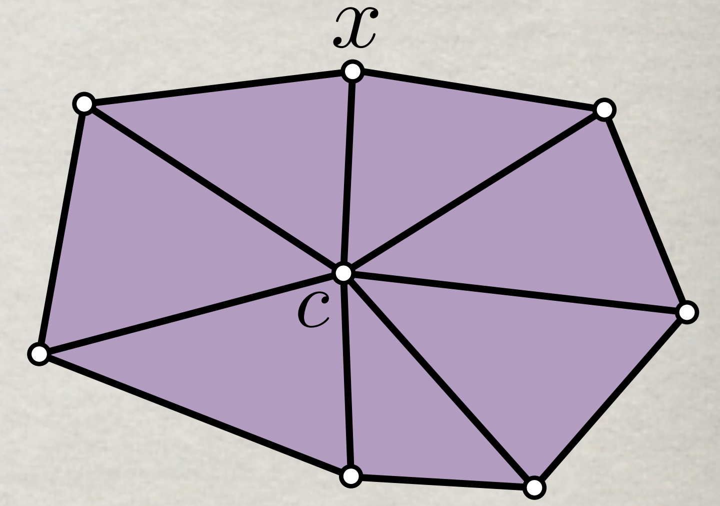
Edge contraction

For arbitrary simplicial complexes, we established that:



$$K \xrightarrow{ab \mapsto c} K'$$

$\not\cong$



implies that

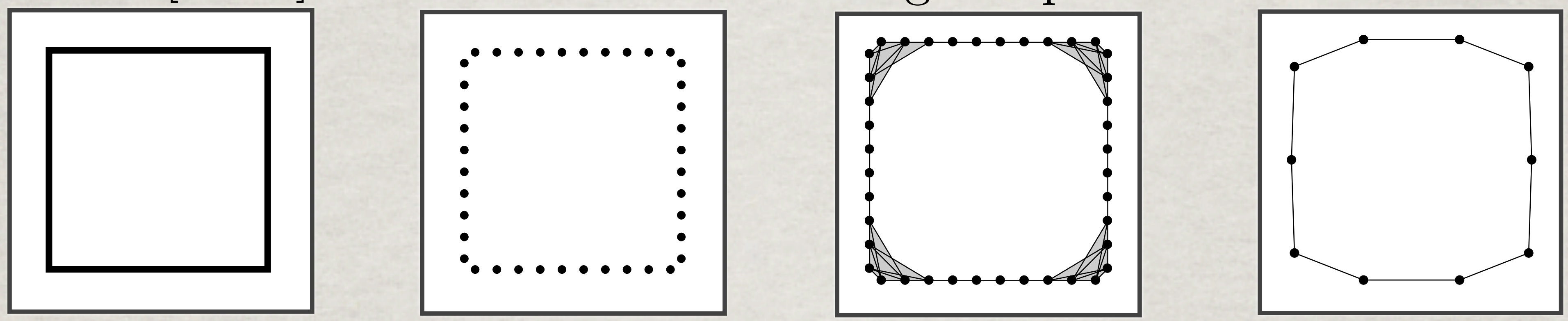
$\mathrm{Lk}\, ab \neq \mathrm{Lk}\, a \cap \mathrm{Lk}\, b$

$\exists$ a blocker of K containing ab

EXPERIMENTS

We keep contracting shortest edge with no blocker through it

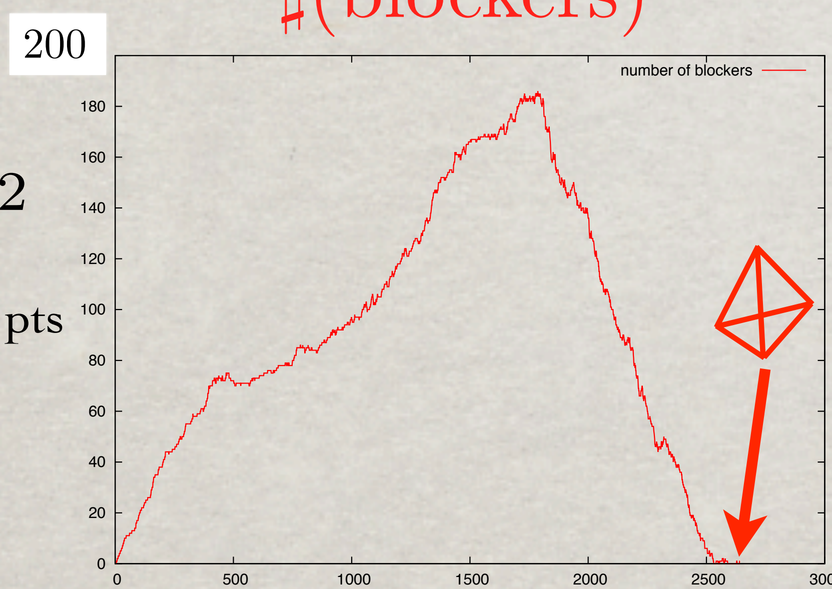
$C_d = \partial[-1,1]^{d+1}$ Point cloud Flag complex $\simeq$

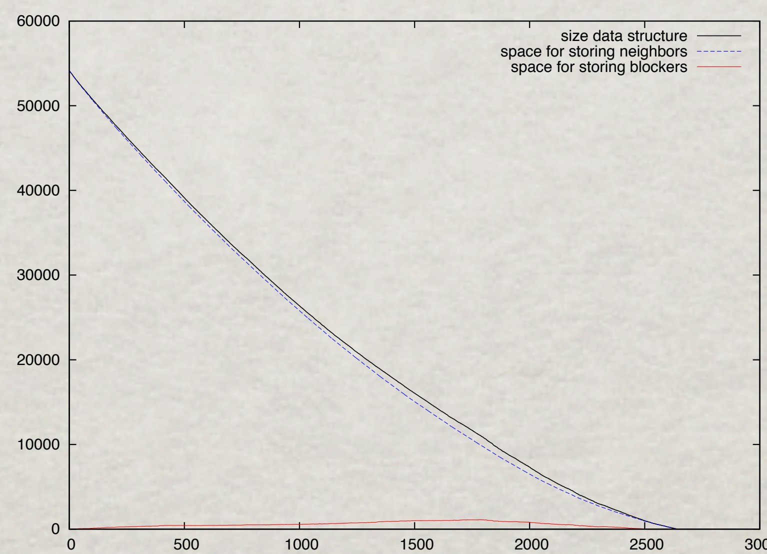


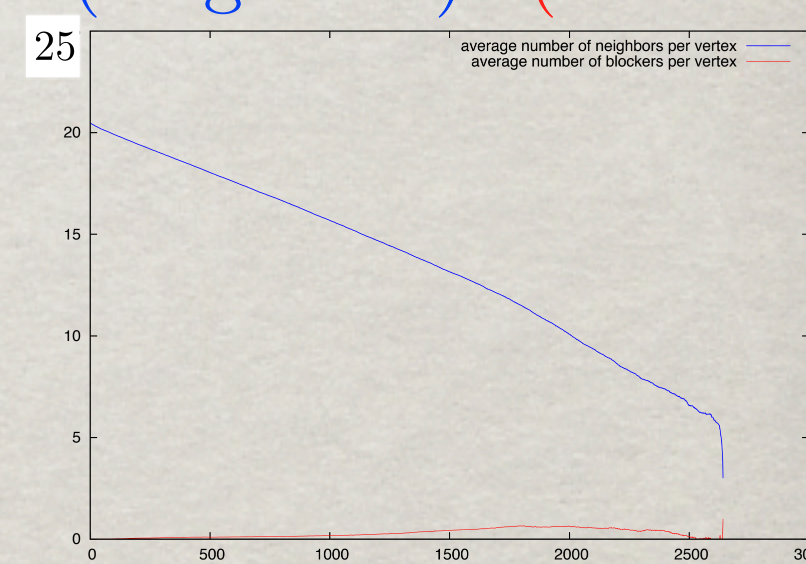
$\sharp$(blockers) size of data structure E(neighbors) E(blockers)

C_2

2646 pts

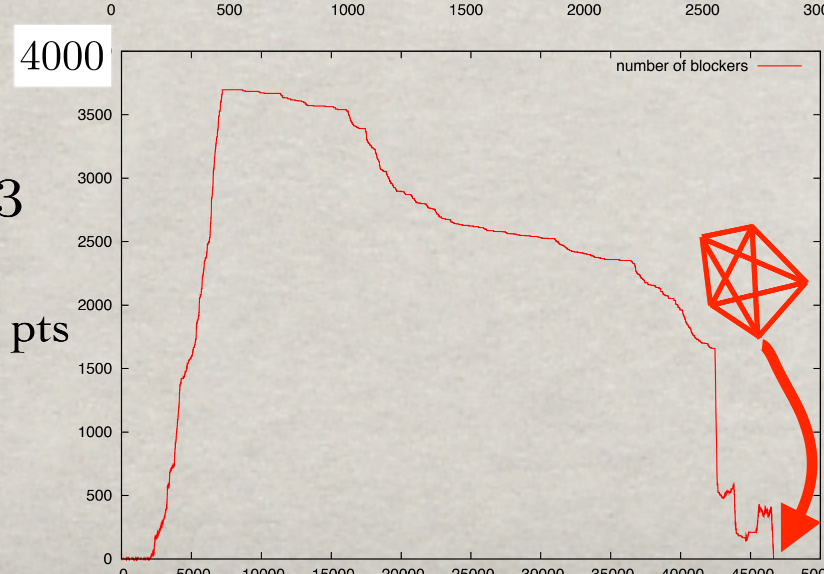


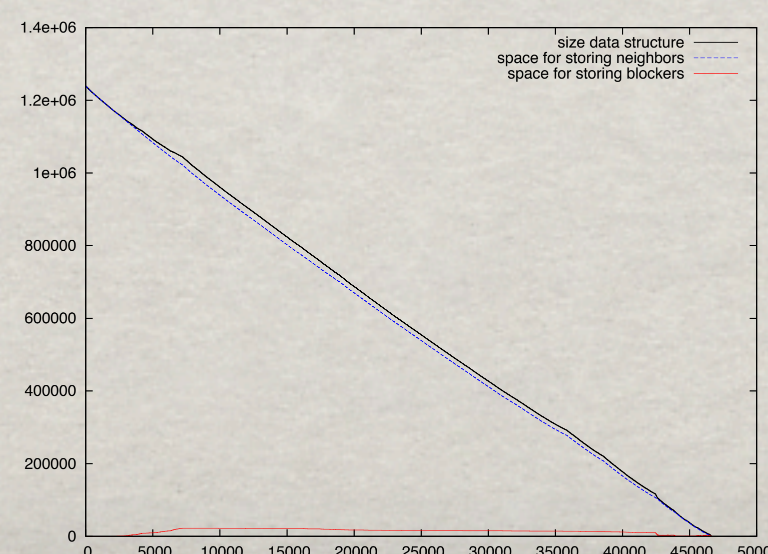


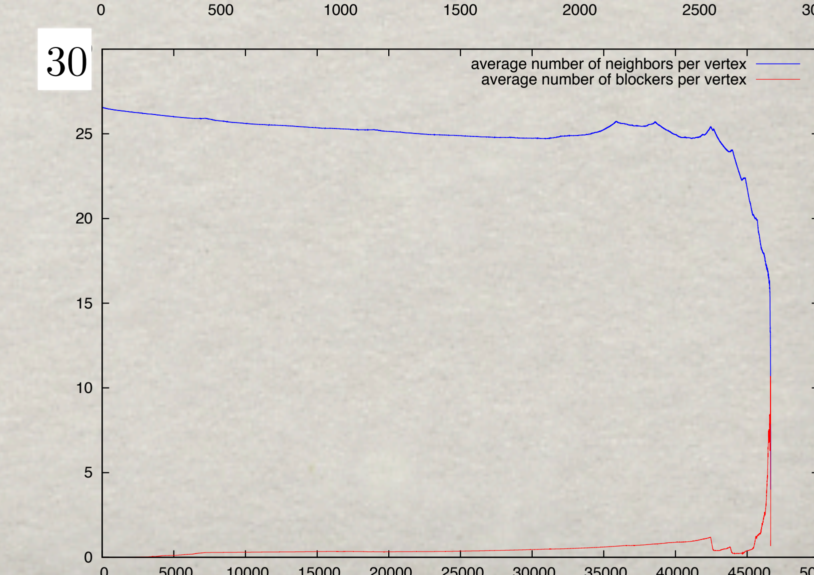


C_3

46656 pts



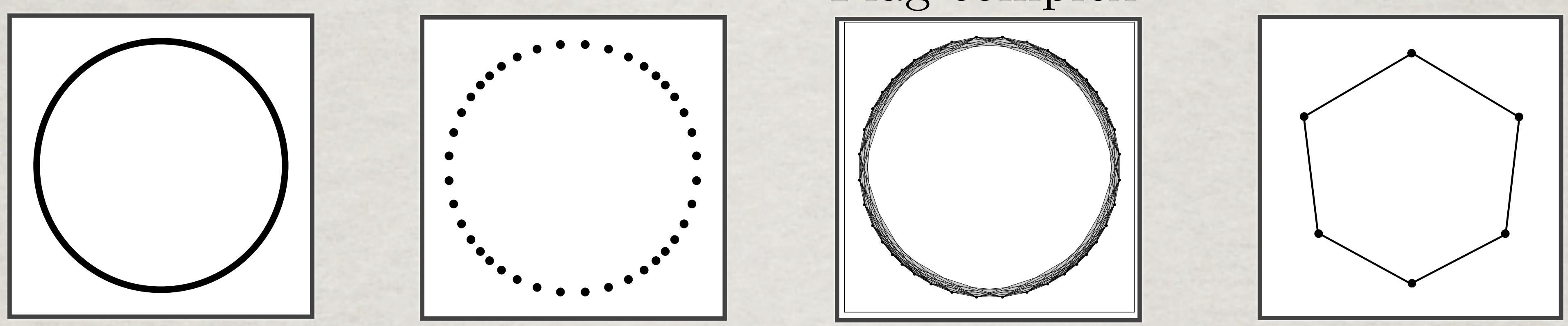


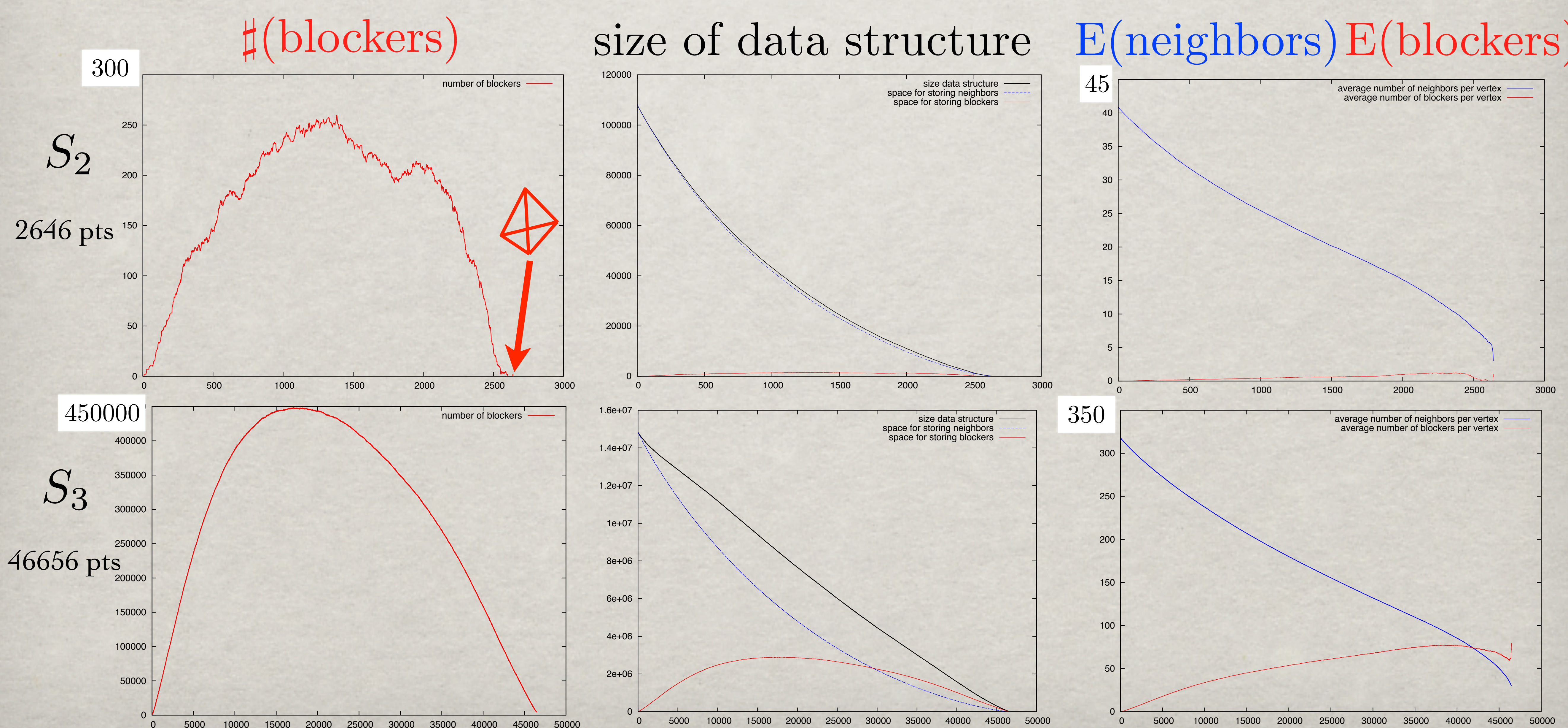


EXPERIMENTS

We keep contracting shortest edge with no blocker through it

$S_d = d$-sphere | Point cloud | Flag complex | $\simeq$





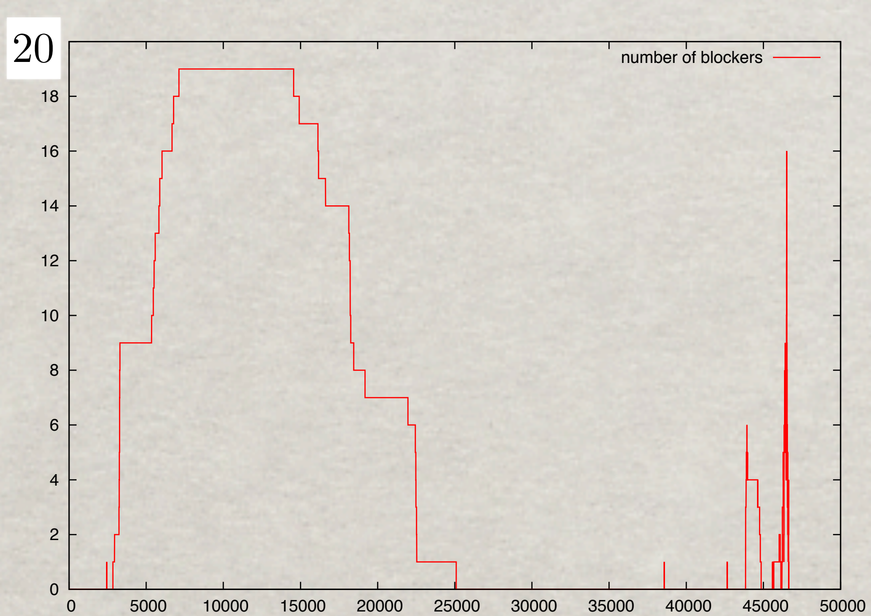
Experiments

$$K = (\,G\,,\ B\,) \xrightarrow{\text{extended anticollapse}} K' = (\,G\,,\ B \setminus \{b\}\,)$$

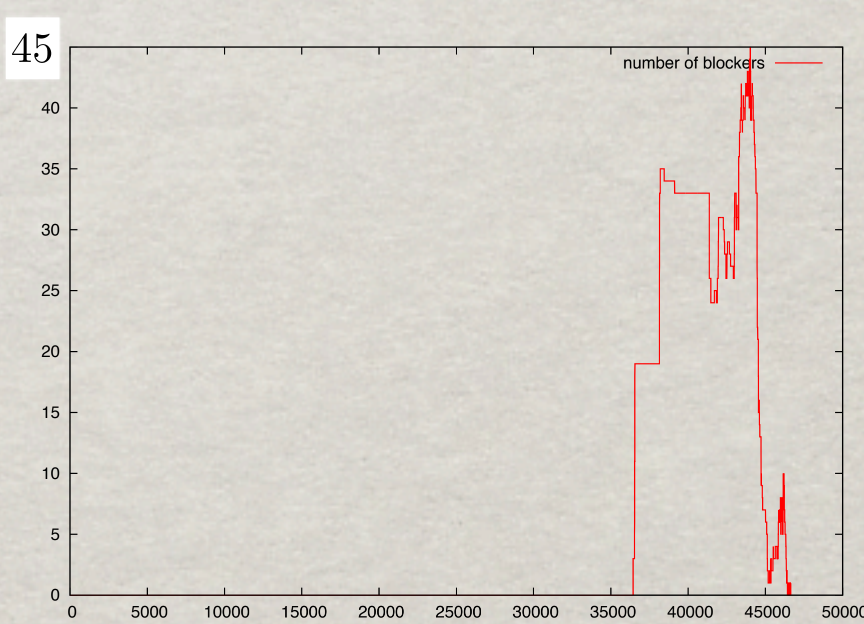
$\simeq$

whenever

$\mathrm{Lk}_{K'}\, b$ is a cone



C3 46656 pts

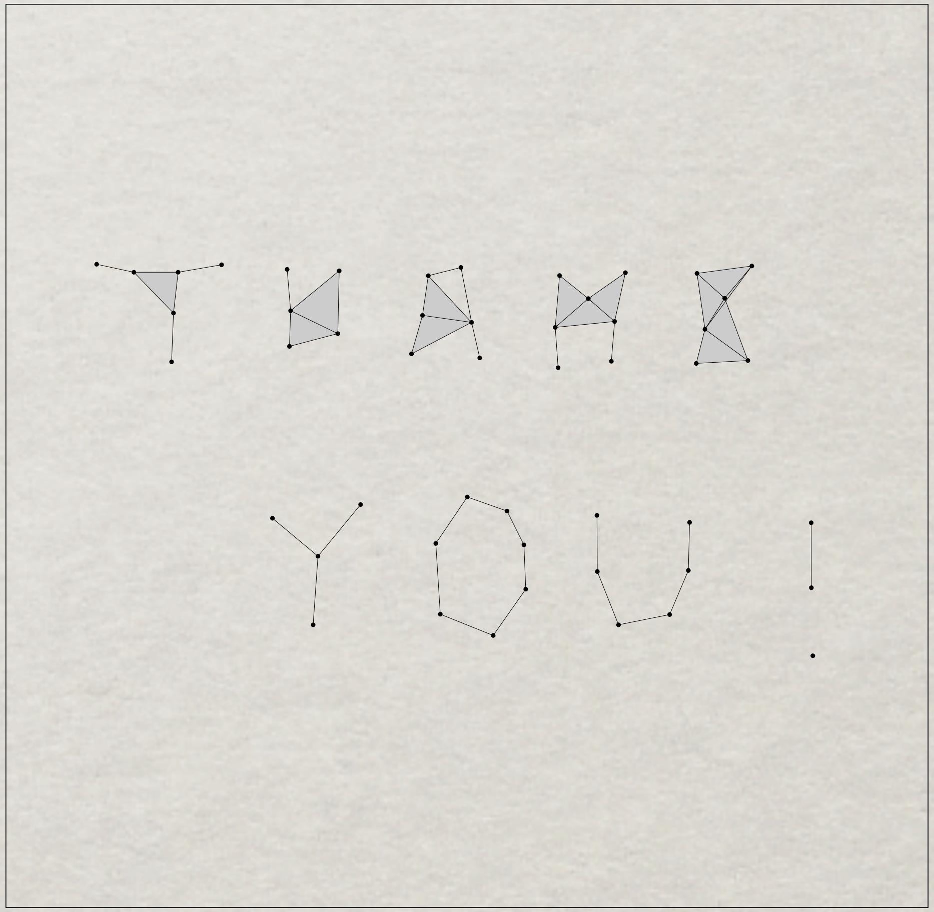


S3 46656 pts

Conclusion

- Promising data structure for encoding high dimensional simplicial complexes.
- Need further invistigations:
  - study other strategies to prioritize contractions besides edge length
  - study other simplification operations besides edge contractions (collapses, ...)


THANK YOU!


THANK YOU!