A linear bound on the Complexity of the Delaunay Triangulation of Points on Polyhedral Surfaces

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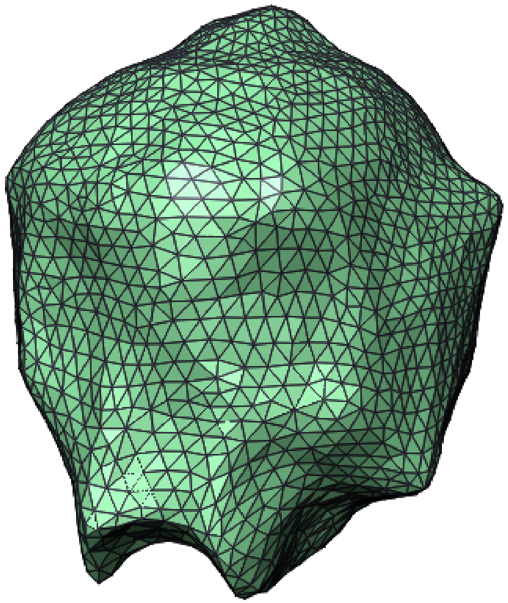
PRISME-INRIA

Introduction

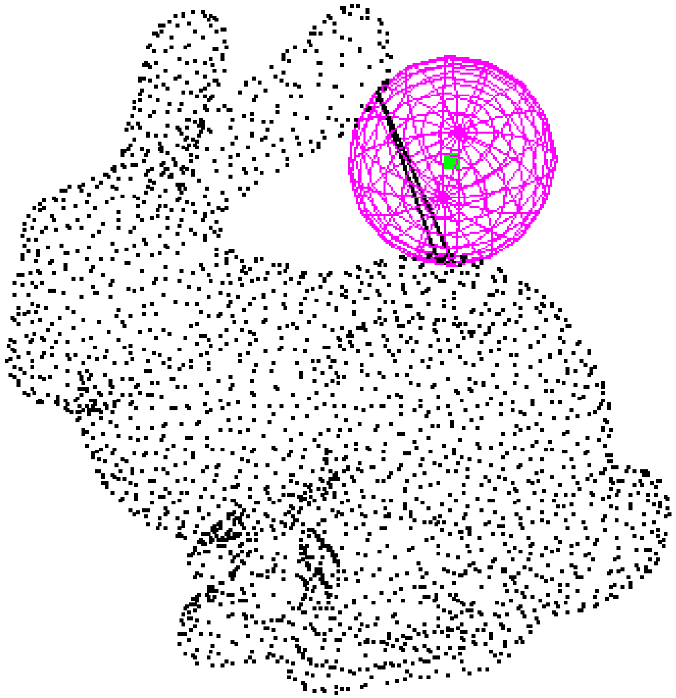
- Applications :
  - mesh generation
  - medial axis approximation
  - surface reconstruction



Question : Complexity of the Delaunay triangulation of points scattered over a surface ?

Complexity of the Delaunay triangulation

- Spheres circumscribing tetrahedra are empty

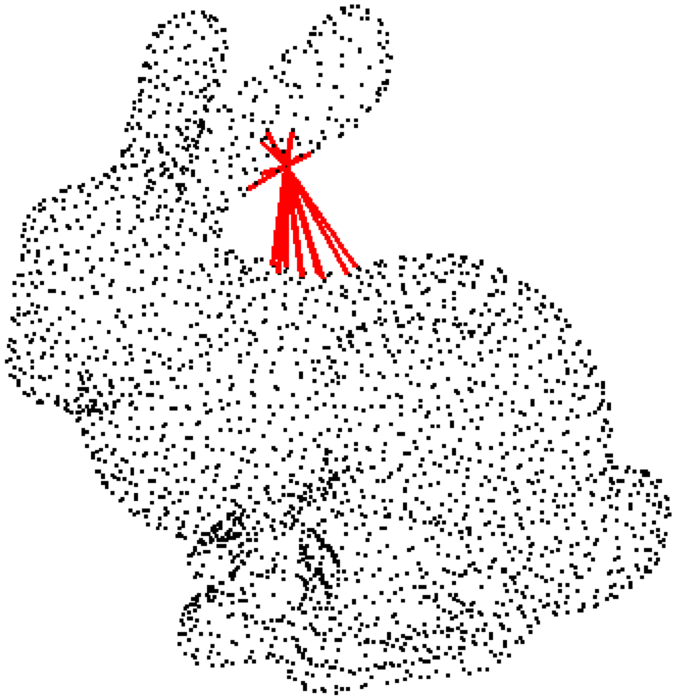


Data points

Convex hull

Complexity of the Delaunay triangulation

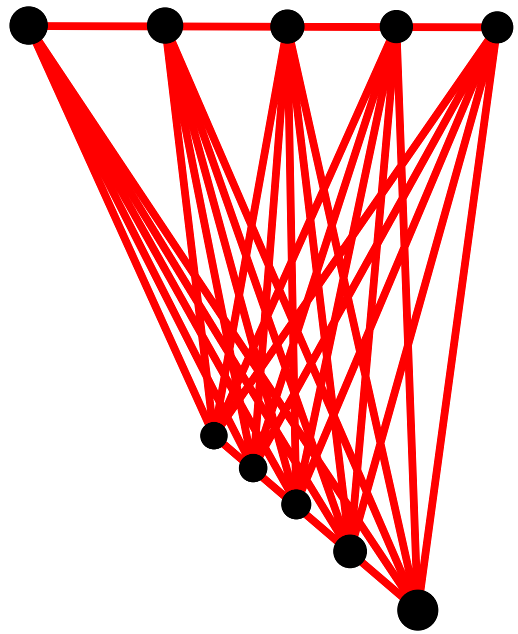
- Complexity = $|$ Edges $| > |$ Tetrahedra $| > |$Triangles$|/4$



Delaunay neighbours

Convex hull

Complexity of the Delaunay triangulation



- For n points, in the worst-case:
  - in $\mathbb{R}^3$, $\Omega(n^2)$

Goal : exhibit practical geometric constraints

for subquadratic / linear bounds.

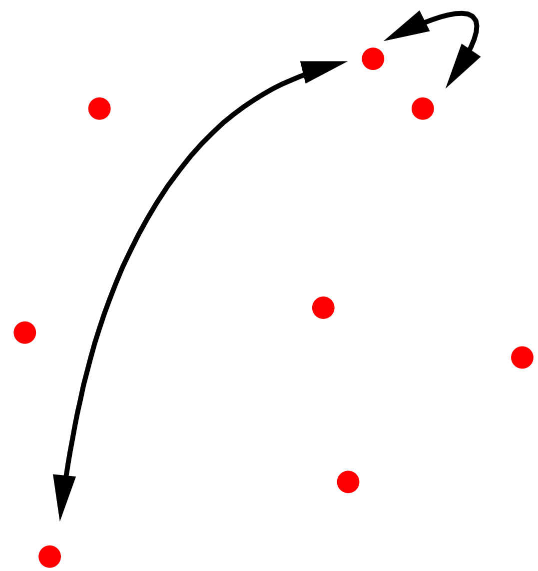
Probabilistic results

- Expected complexity for n random points on
  - a ball : $\Theta(n)$ [Dwyer 1993]
  - a convex polytope : $\Theta(n)$ [Golin & Na 2000]
  - a polytope : $O(n \log^4 n)$ [Golin & Na 2002]

Deterministic results

- Wrt spread : $O(\text{spread}^3)$ [Erickson 2002]

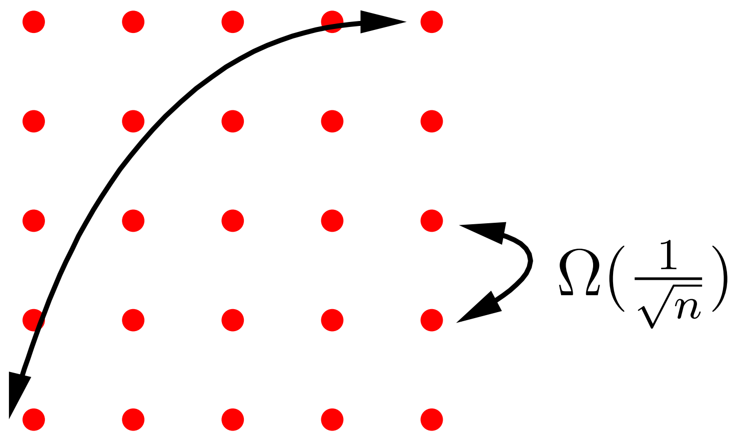
$$\text{Spread} = \frac{\text{largest interpoint distance}}{\text{smallest interpoint distance}}$$



Deterministic results

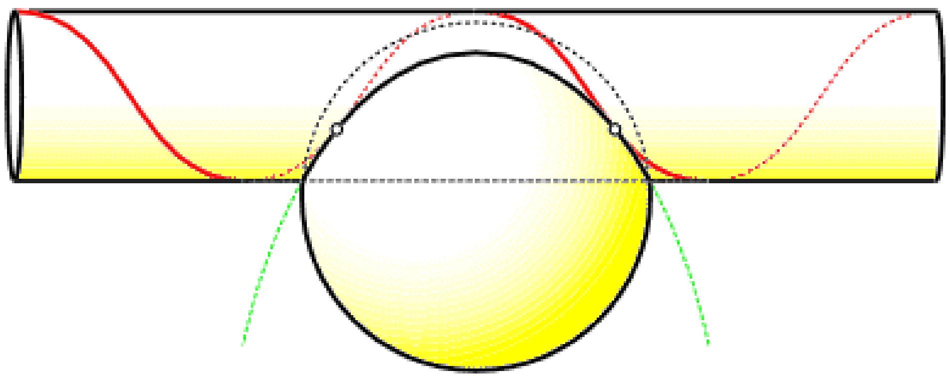
- Wrt spread : $O(\text{Spread}^3)$ [Erickson 2002]
  - surfaces sampled with spread $O(\sqrt{n})$: $O(n\sqrt{n})$

$$\text{Spread} = \frac{\text{largest interpoint distance}}{\text{smallest interpoint distance}}$$
$$= O(\sqrt{n})$$



Deterministic results

- Wrt spread : $O(\text{Spread}^3)$ [Erickson 2002]
  - surfaces sampled with spread $O(\sqrt{n})$: $O(n\sqrt{n})$
  - Well-sampled cylinder : $\Omega(n\sqrt{n})$



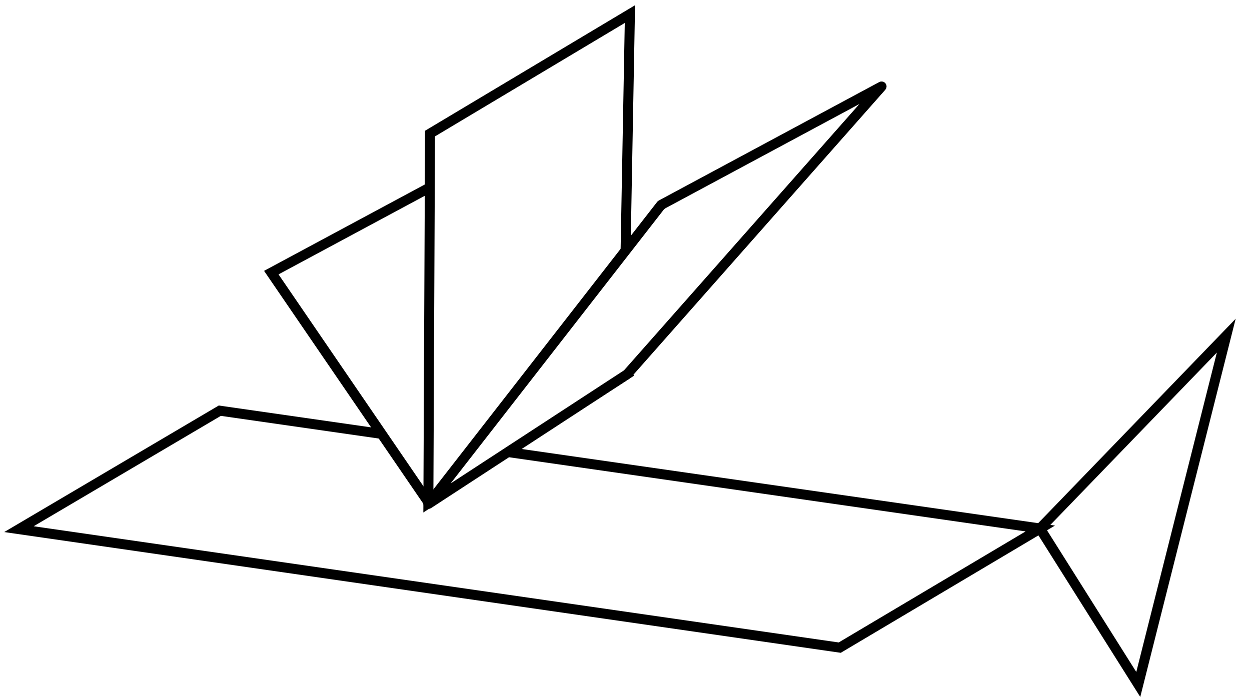
Our main result

For points distributed on a polyedral surface in $\mathbb{R}^3$: the Delaunay triangulation is linear

- Deterministic result
  - polyedral surface
  - sampling condition
  - proof

Polyedral surface

- Polyedral surface = Finite collection of facets that form a pur piece-wise linear complex
- Facet = bounded polygon

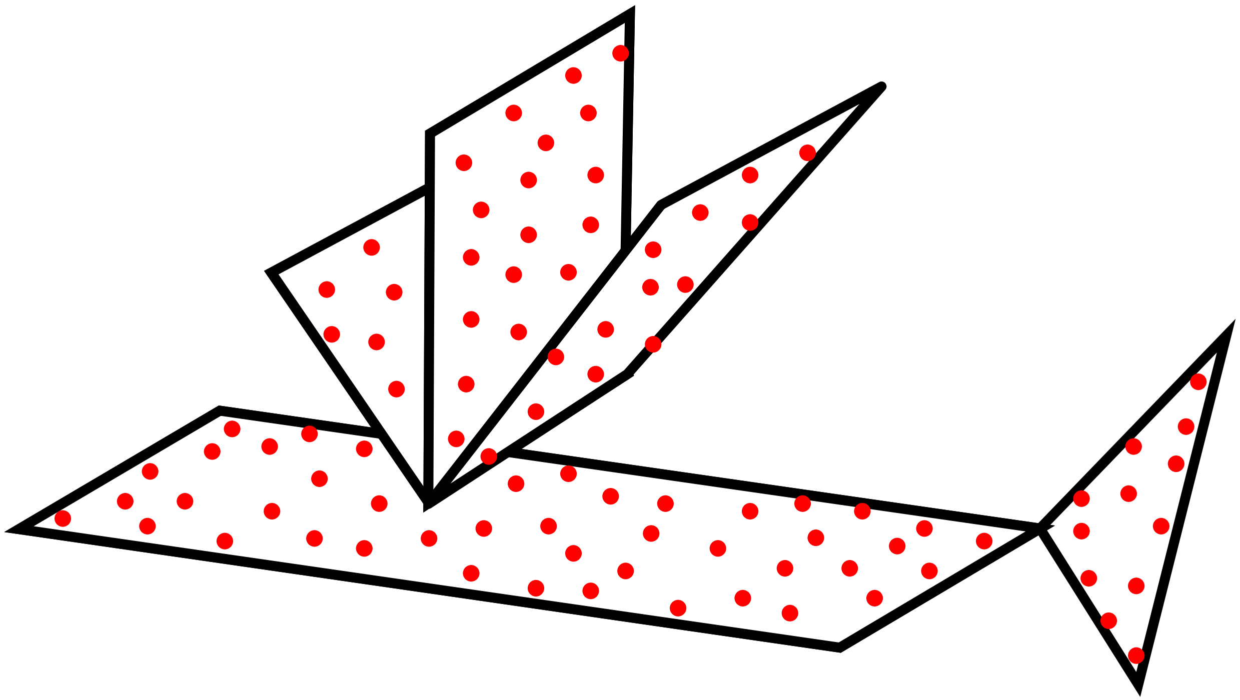


Sampling condition

- (ε, κ)-sample E :

1.

2.

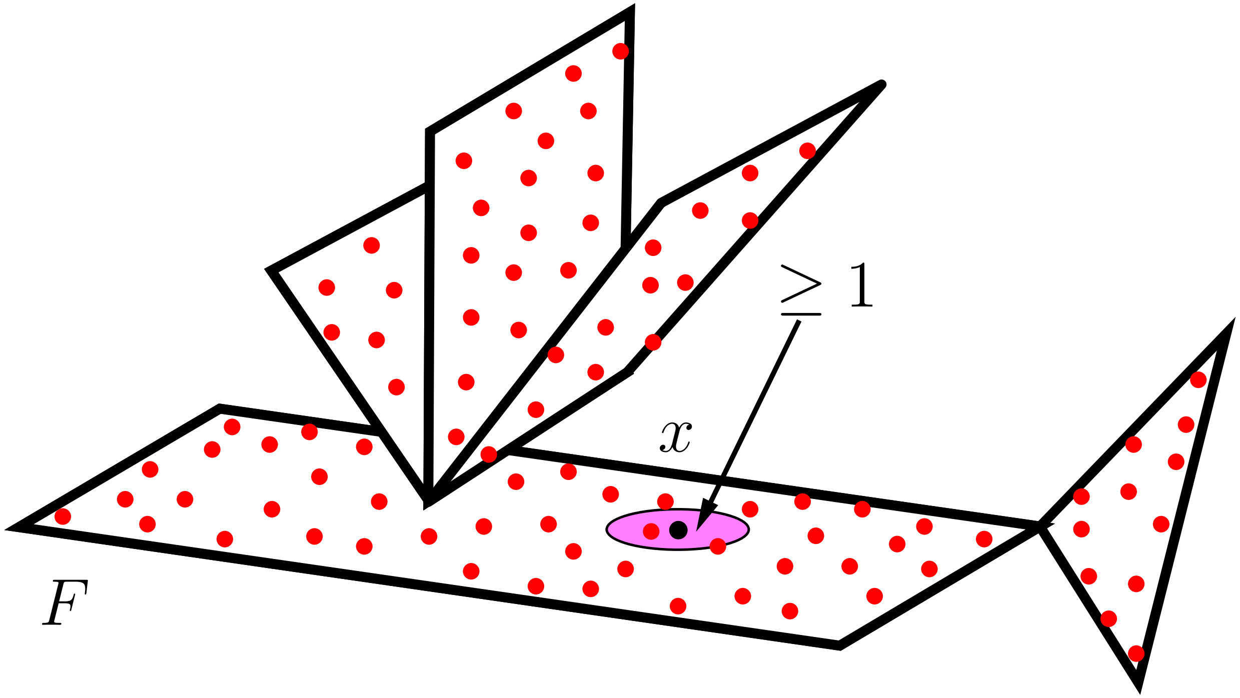


Sampling condition

- (ε, κ)-sample E :

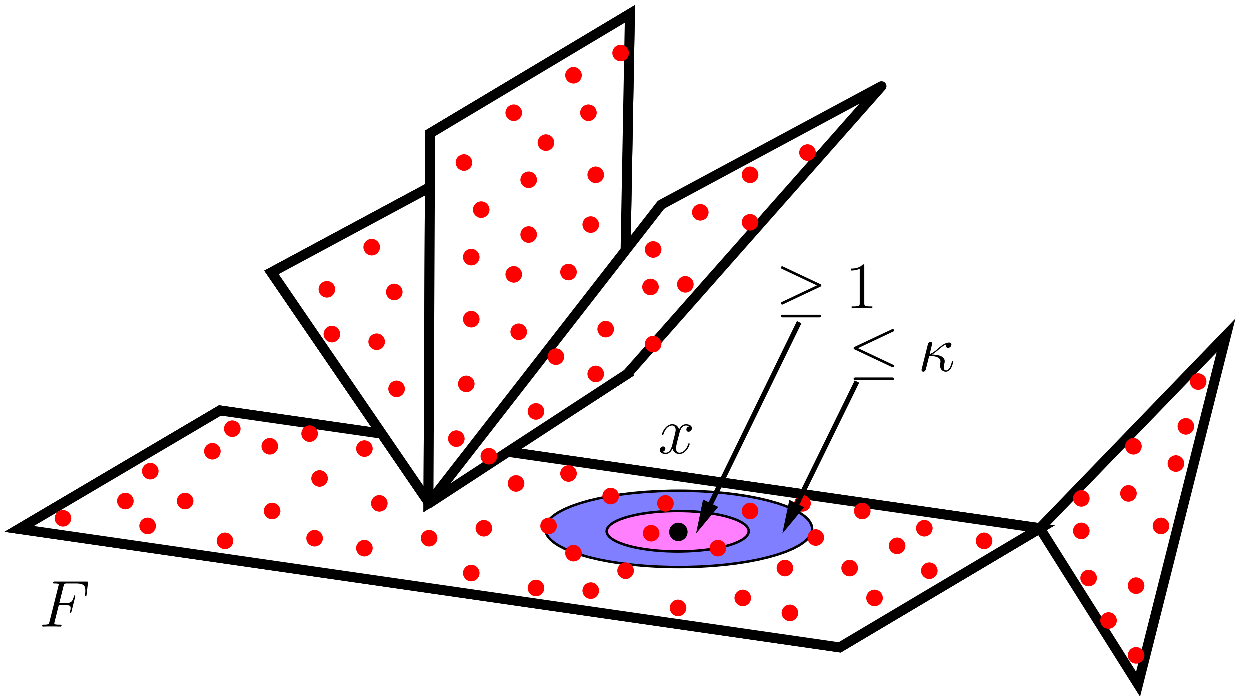
1. $\forall x \in F$, $B(x, \varepsilon)$ encloses at least one point of $E \cap F$

2.



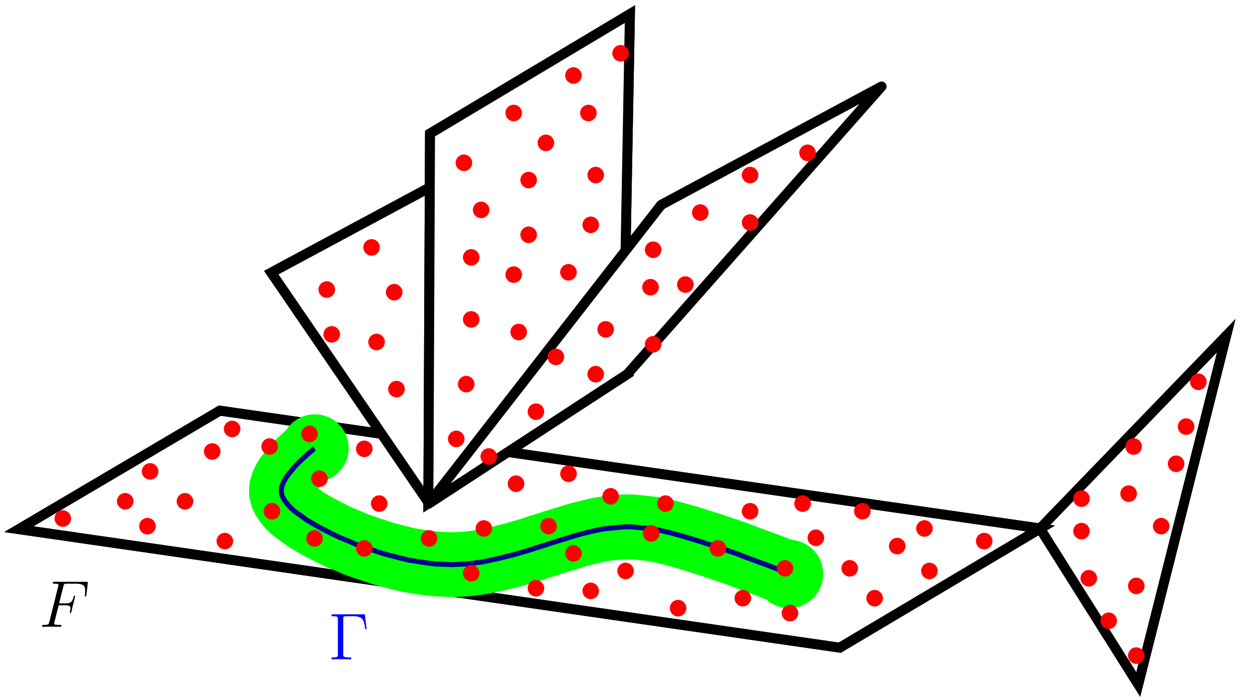
Sampling condition

- (ε, κ)-sample E :
  1. $\forall x \in F$, $B(x, \varepsilon)$ encloses at least one point of $E \cap F$
  2. $\forall x \in F$, $B(x, 2\varepsilon)$ encloses at most κ points of $E \cap F$



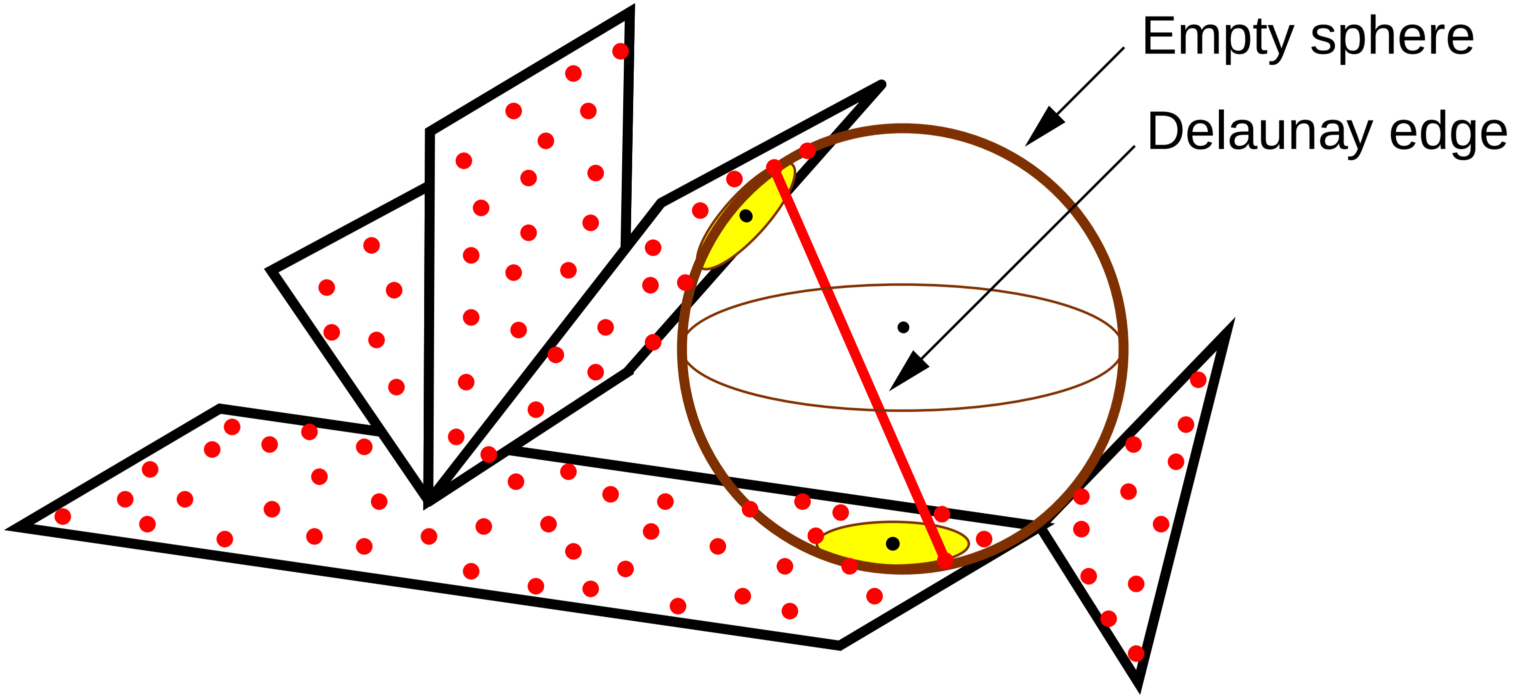
Sampling condition

- $n = \Theta\left(\frac{1}{\varepsilon^2}\right)$
- $n(\Gamma \oplus \varepsilon) = O(\,length(\Gamma) \times \sqrt{n}\,)$



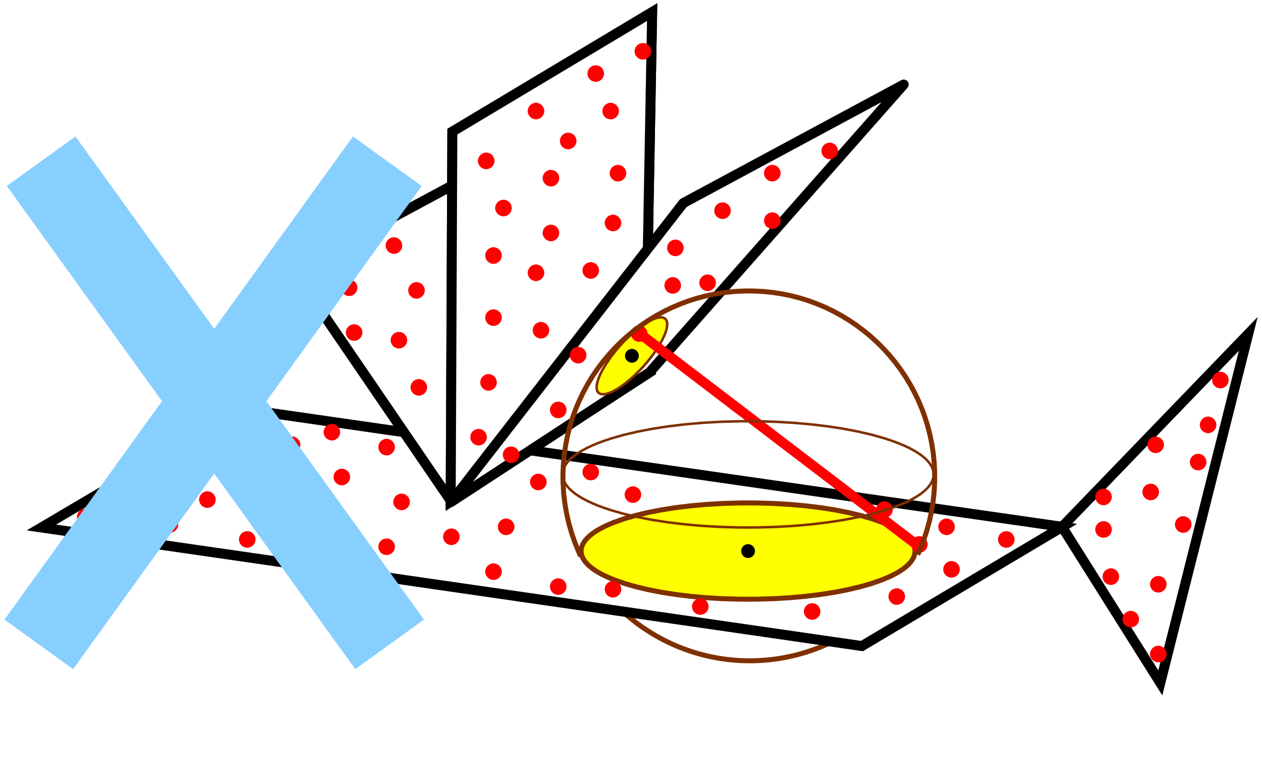
Delaunay triangulation

- Assumptions : (ε, κ)-sample of a polyedral surface
- Proof : Count Delaunay edges



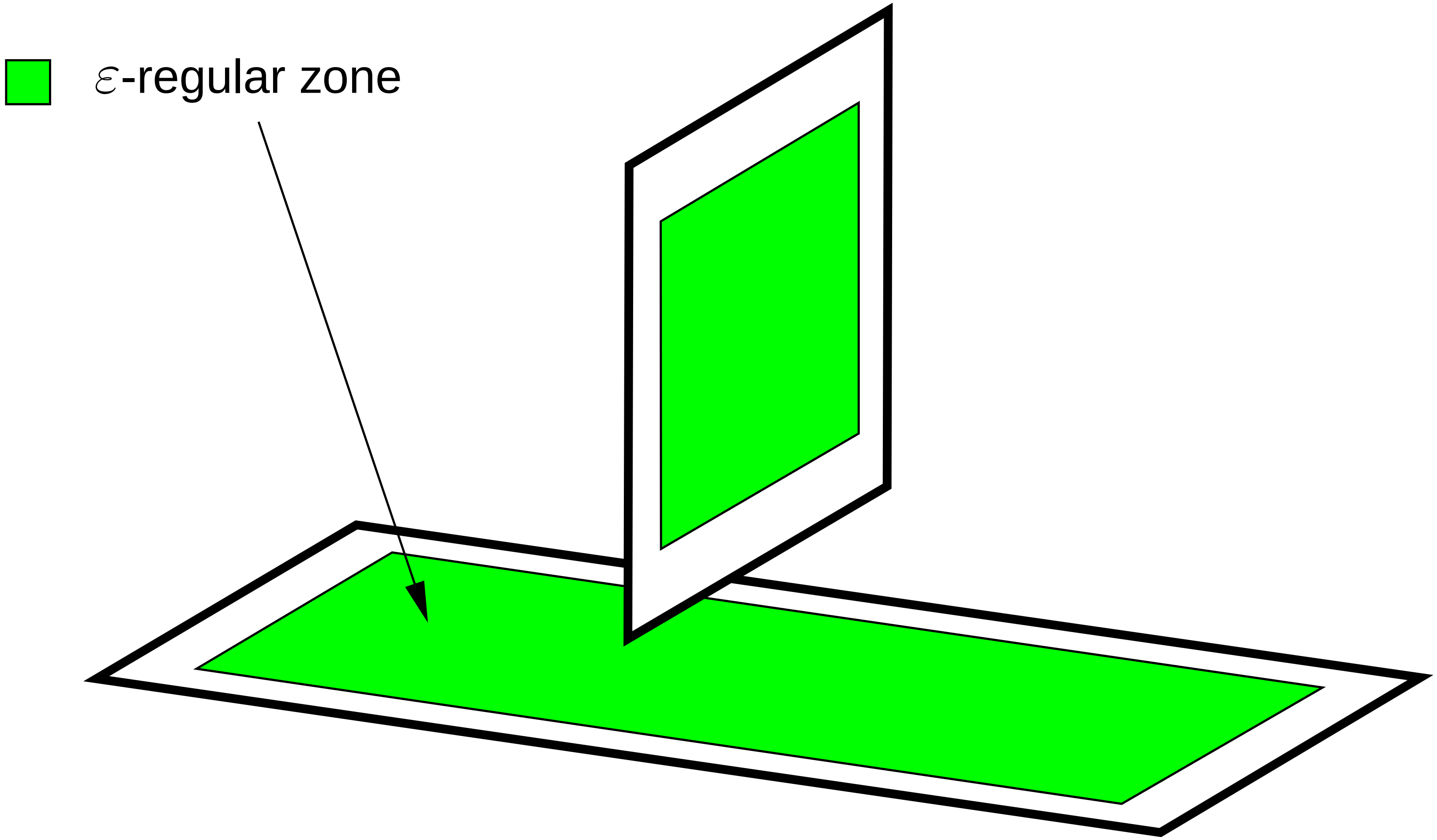
Proof

- Count Delaunay edges



Counting Delaunay edges

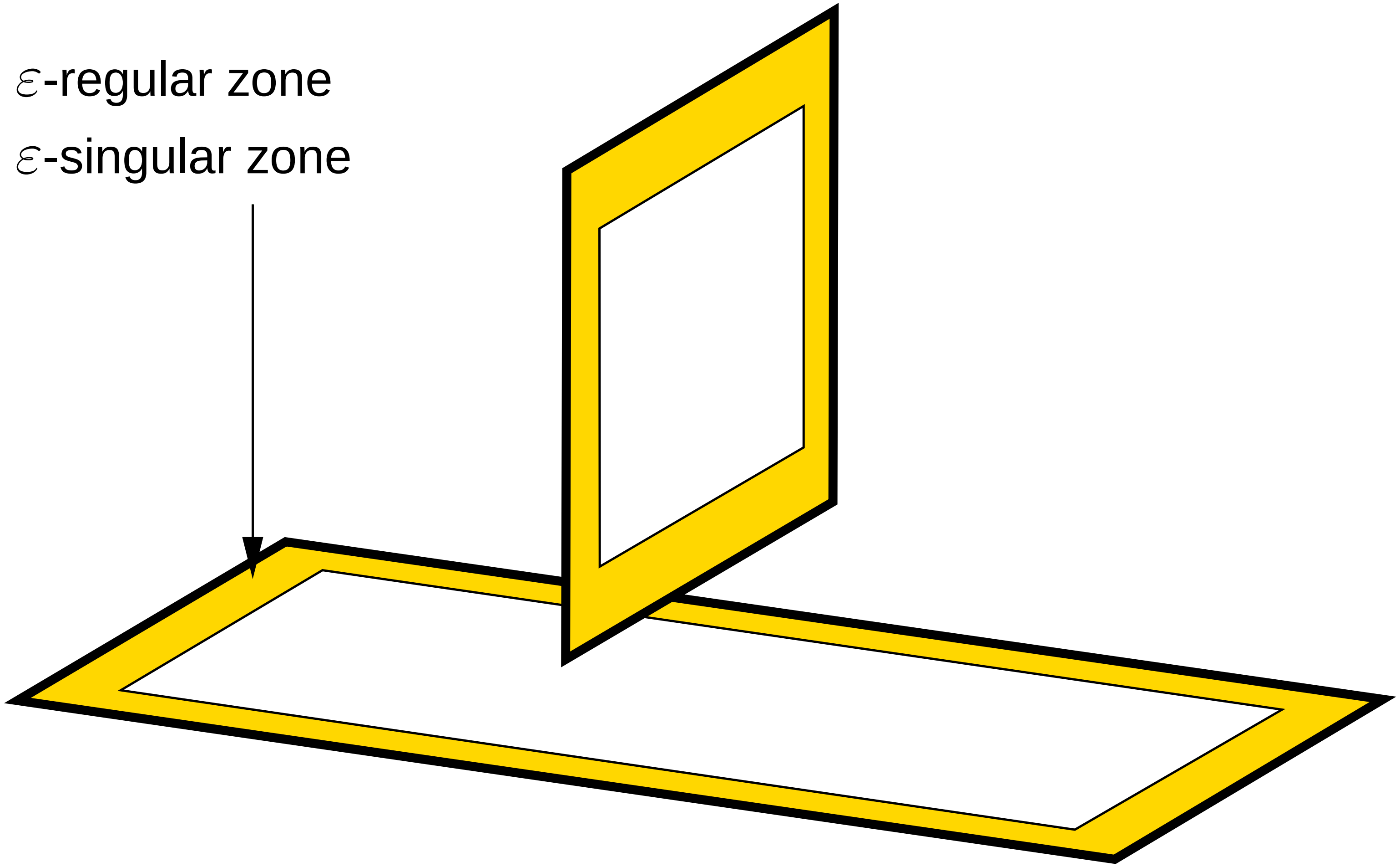
- 2 zones on the surface



Counting Delaunay edges

- 2 zones on the surface

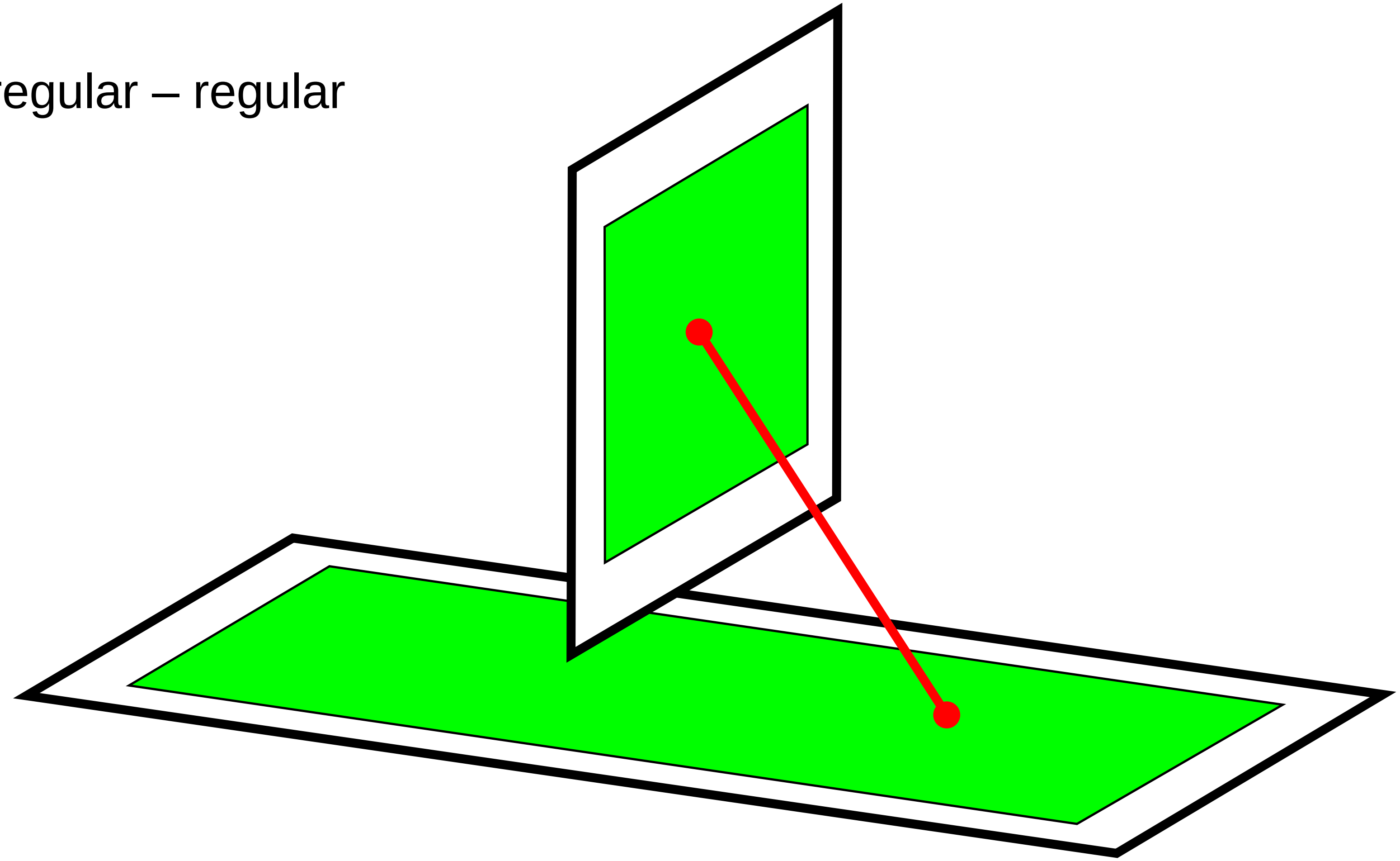
  - ε-regular zone
  - ε-singular zone



Counting Delaunay edges

- 3 types of edges

  ① regular – regular



Counting Delaunay edges

- 3 types of edges

  ① regular – regular
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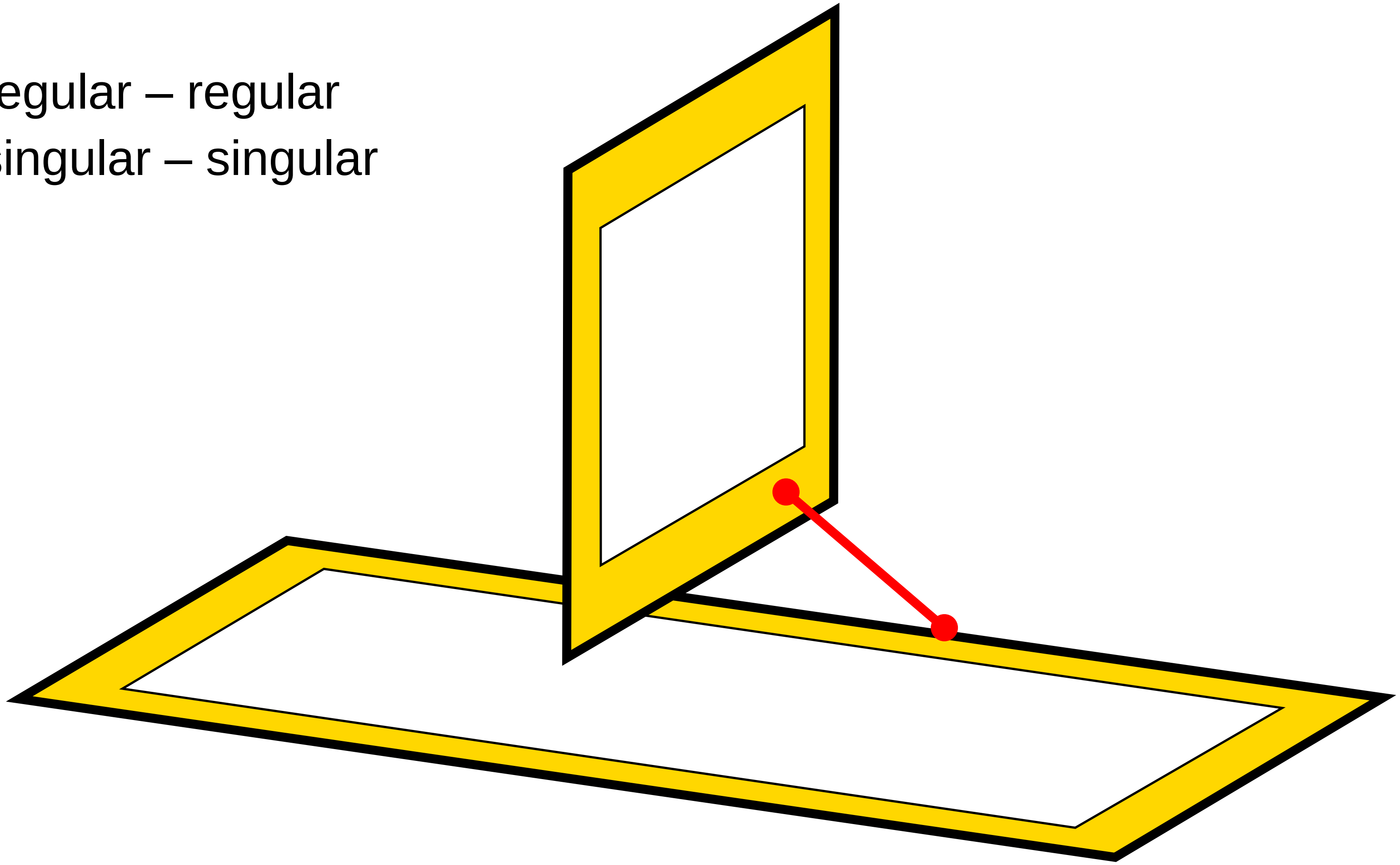
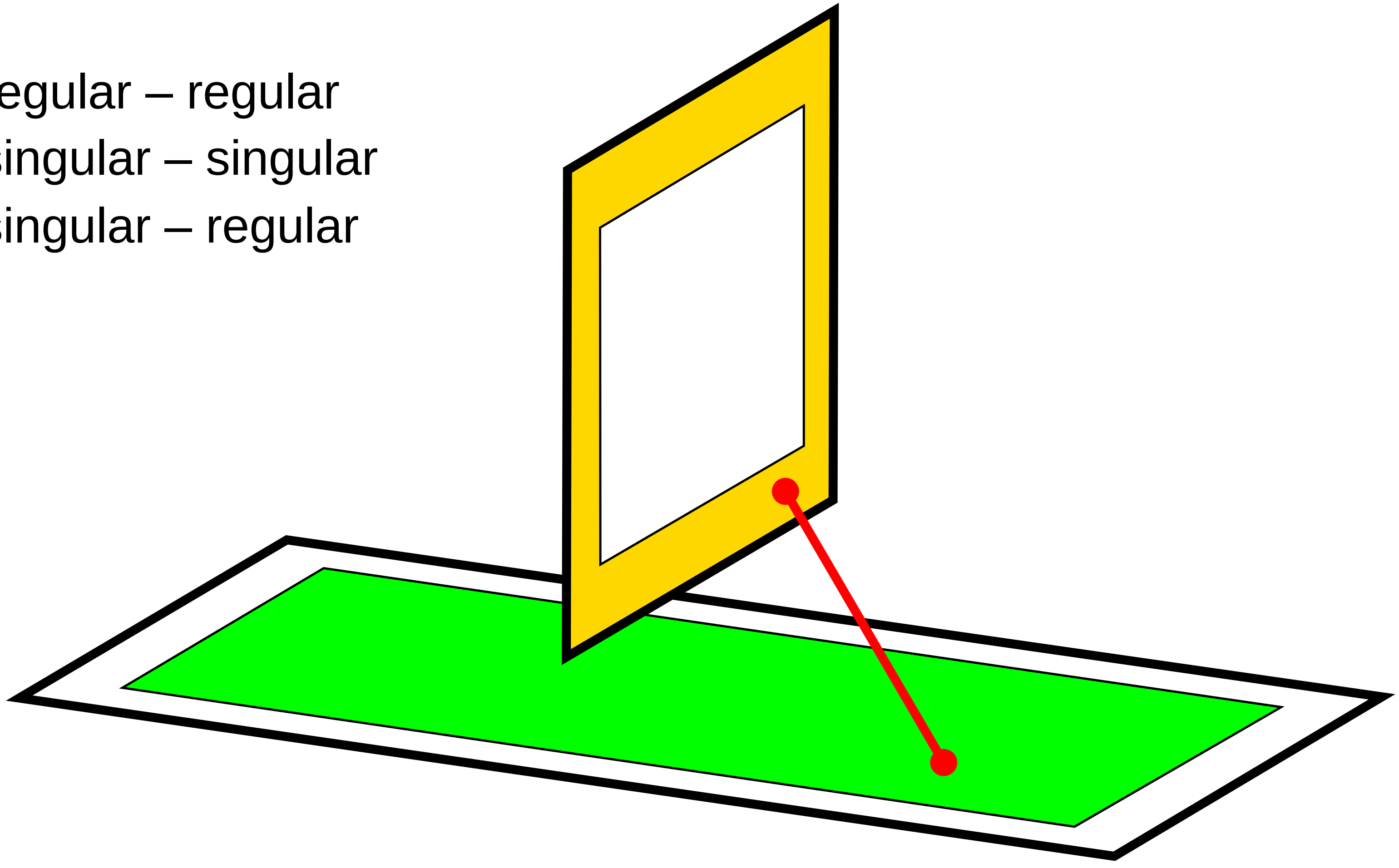


Counting Delaunay edges

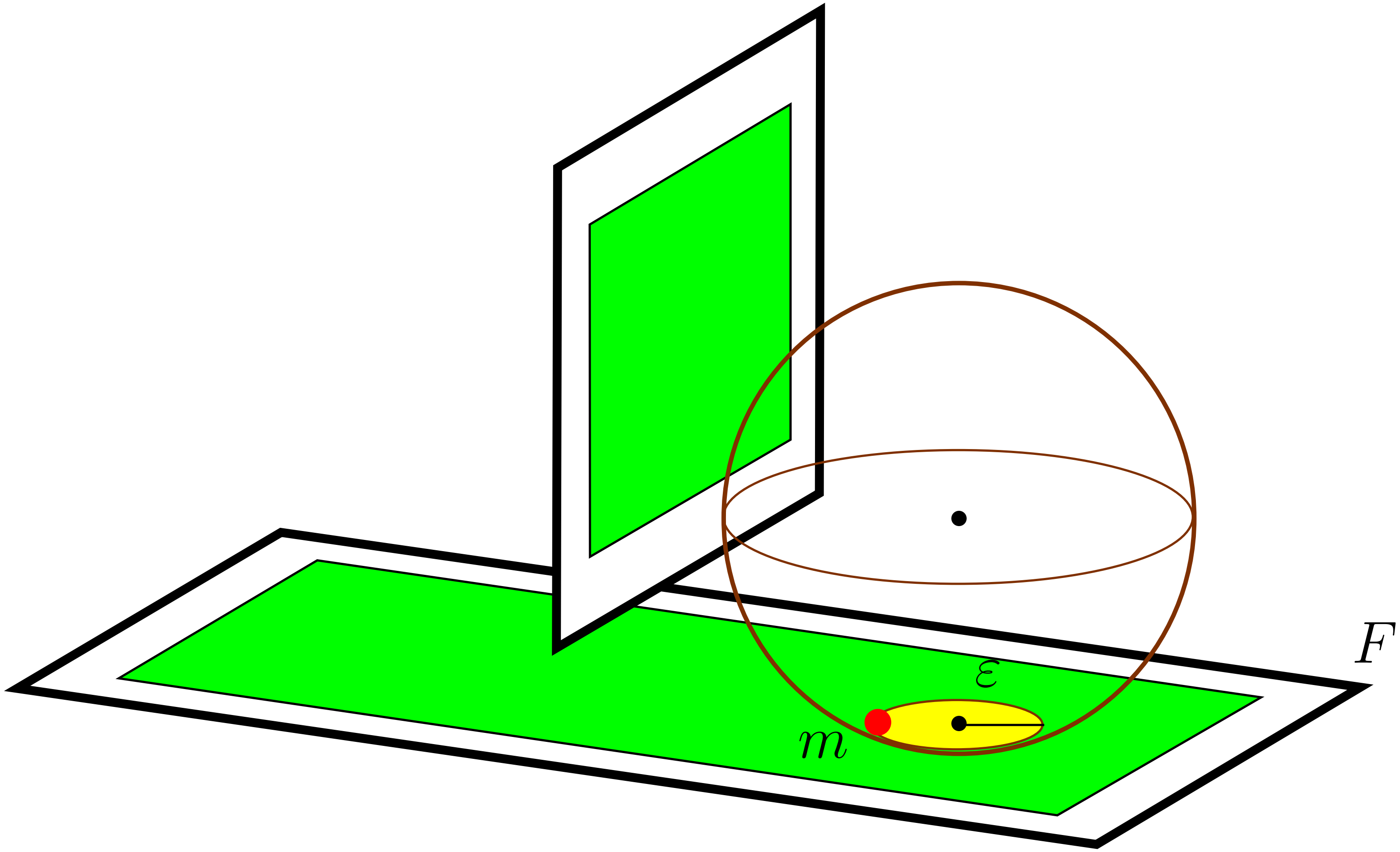
- 3 types of edges

  ① regular – regular
  ② singular – singular
  ③ singular – regular



Regular - Regular

- A sample point has at most κ neighbours in its own facet



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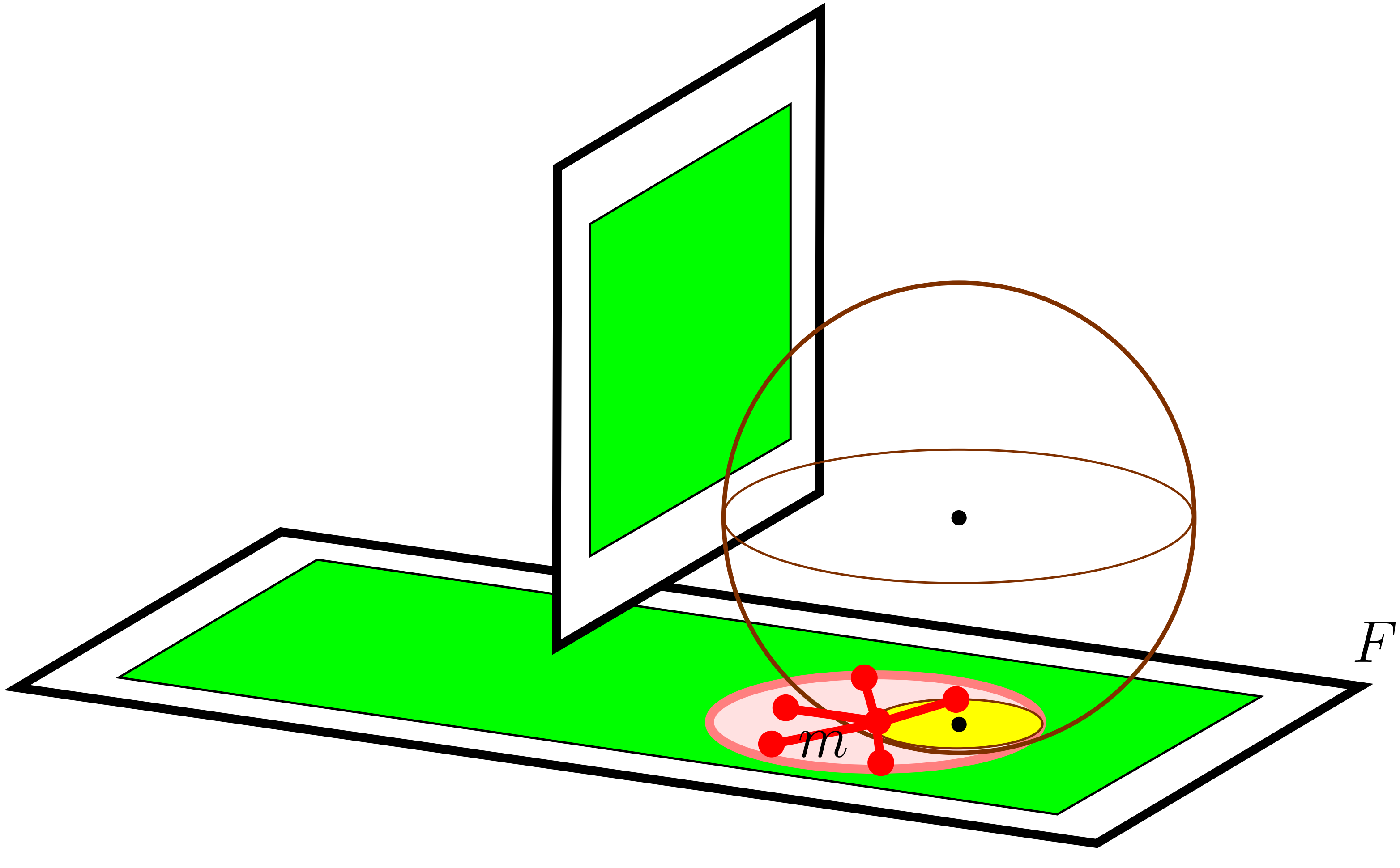

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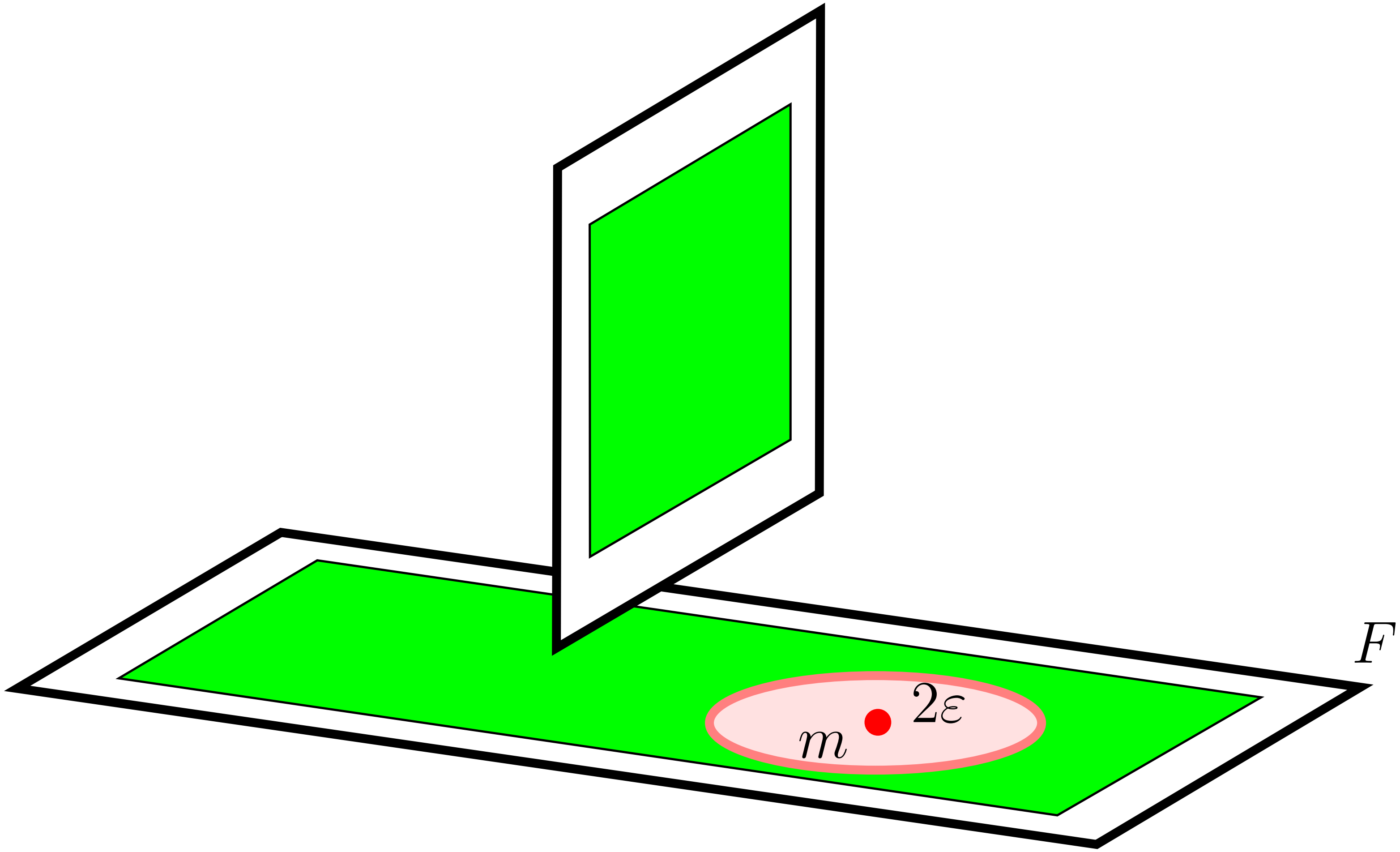
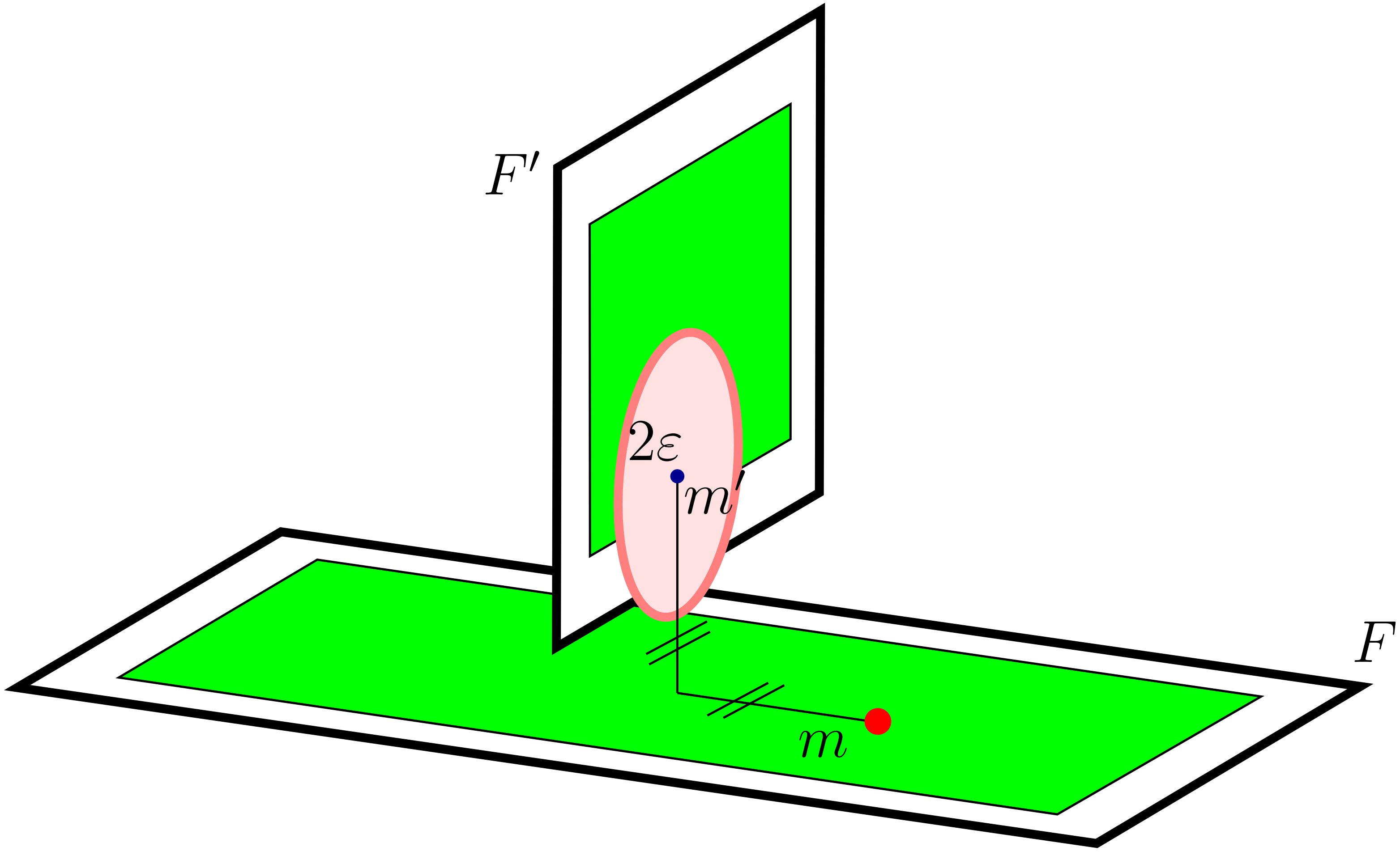


Regular - Regular

- A sample point has at most κ neighbours in any facet



Regular - Regular

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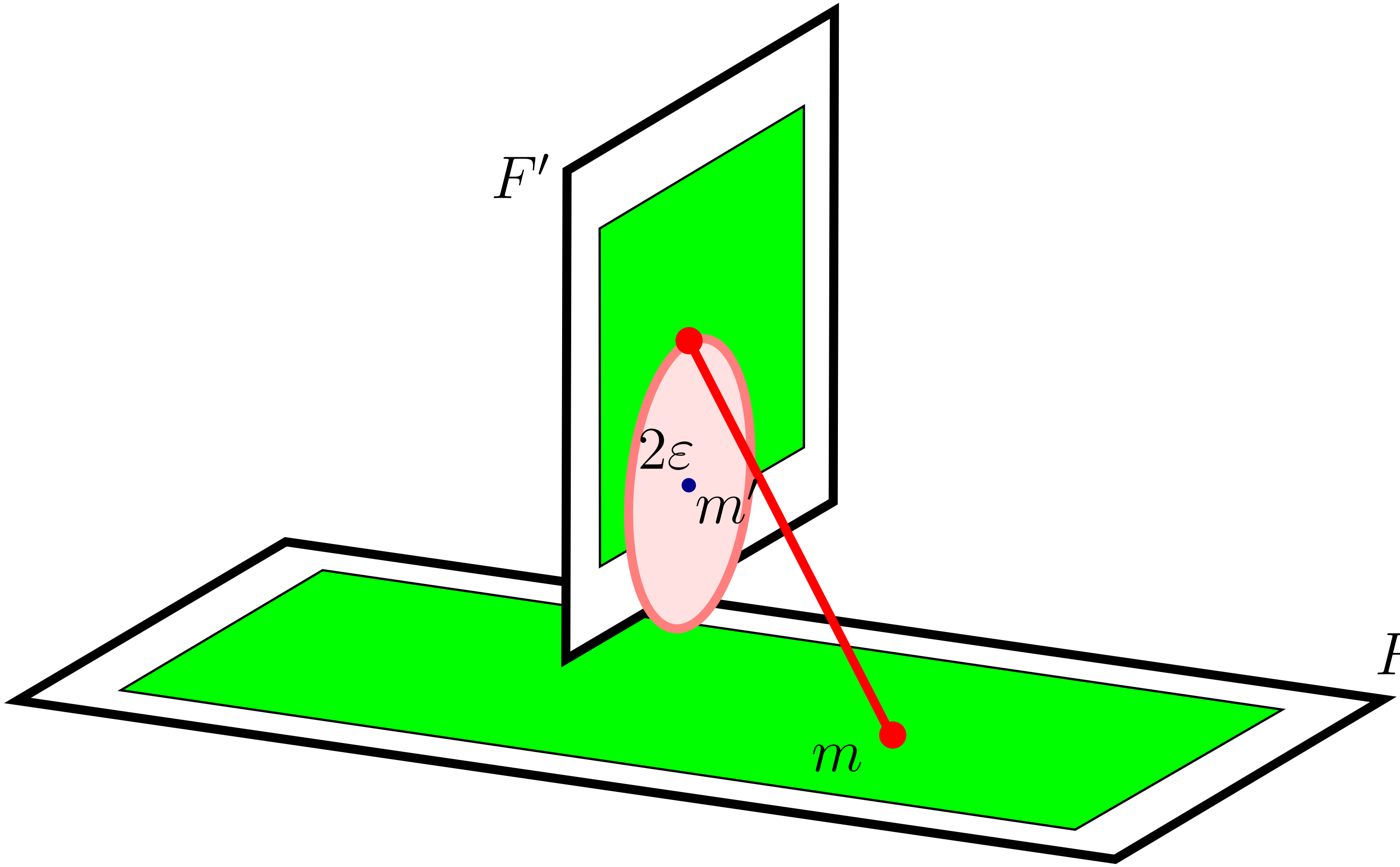

Regular - Regular

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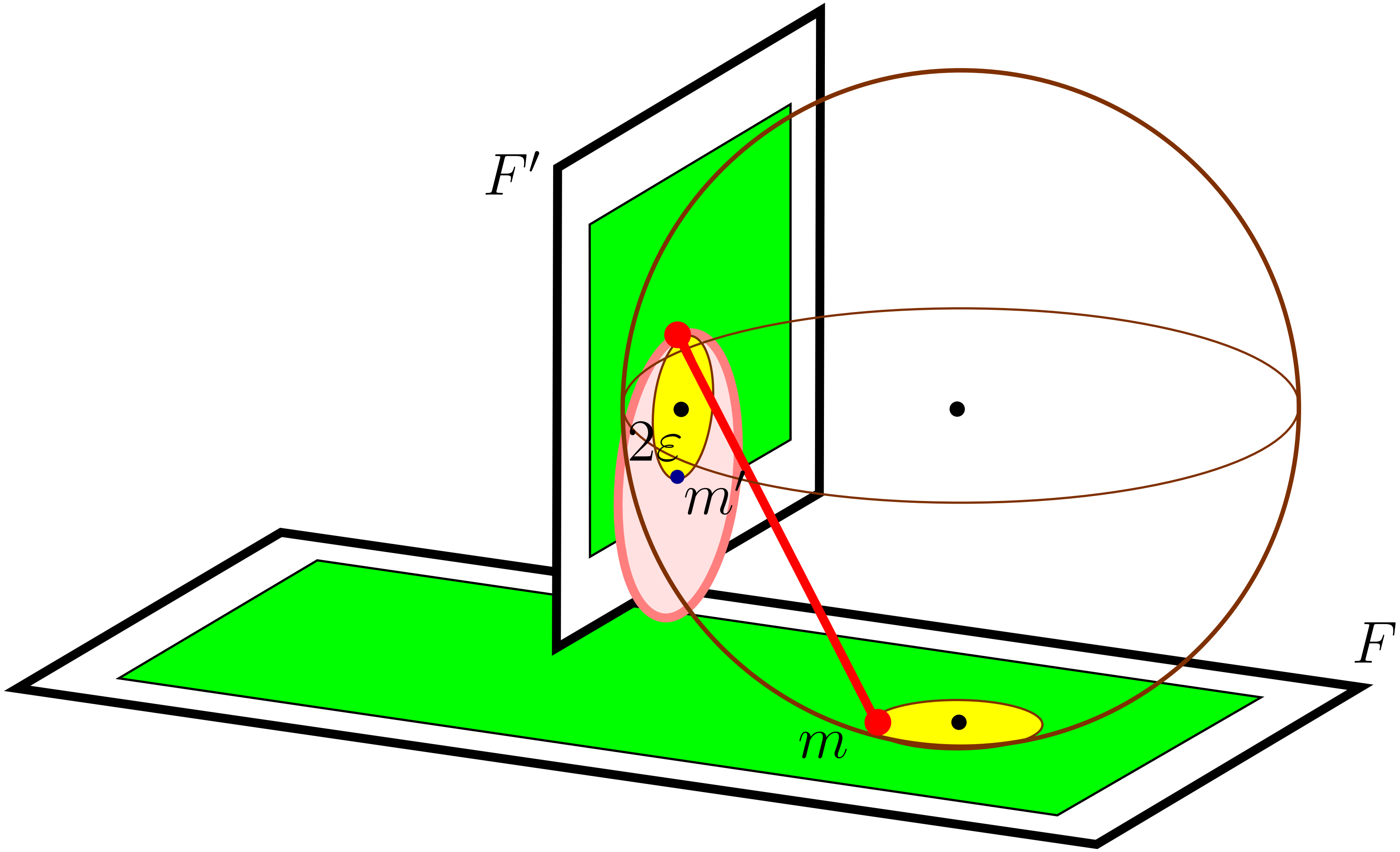
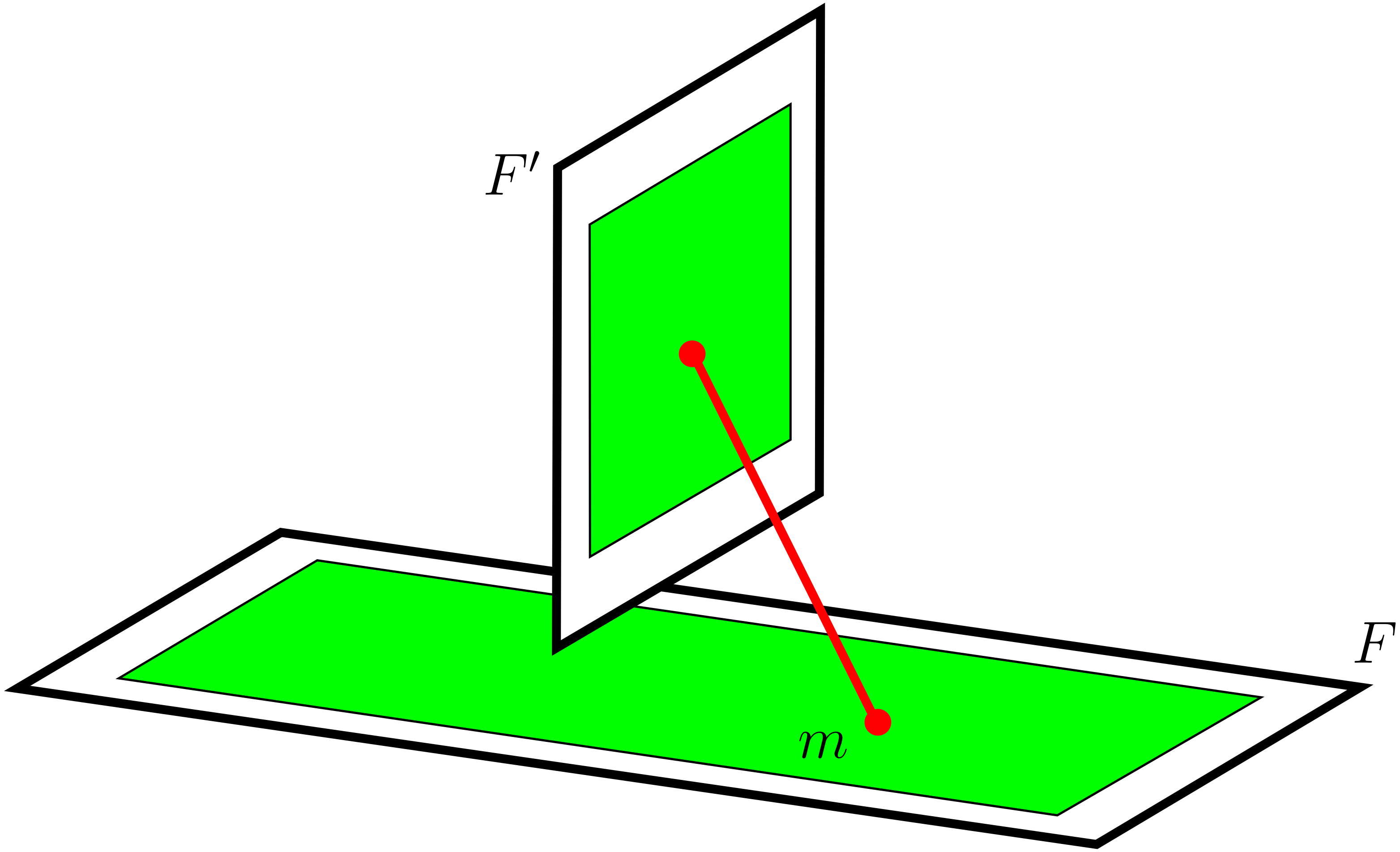


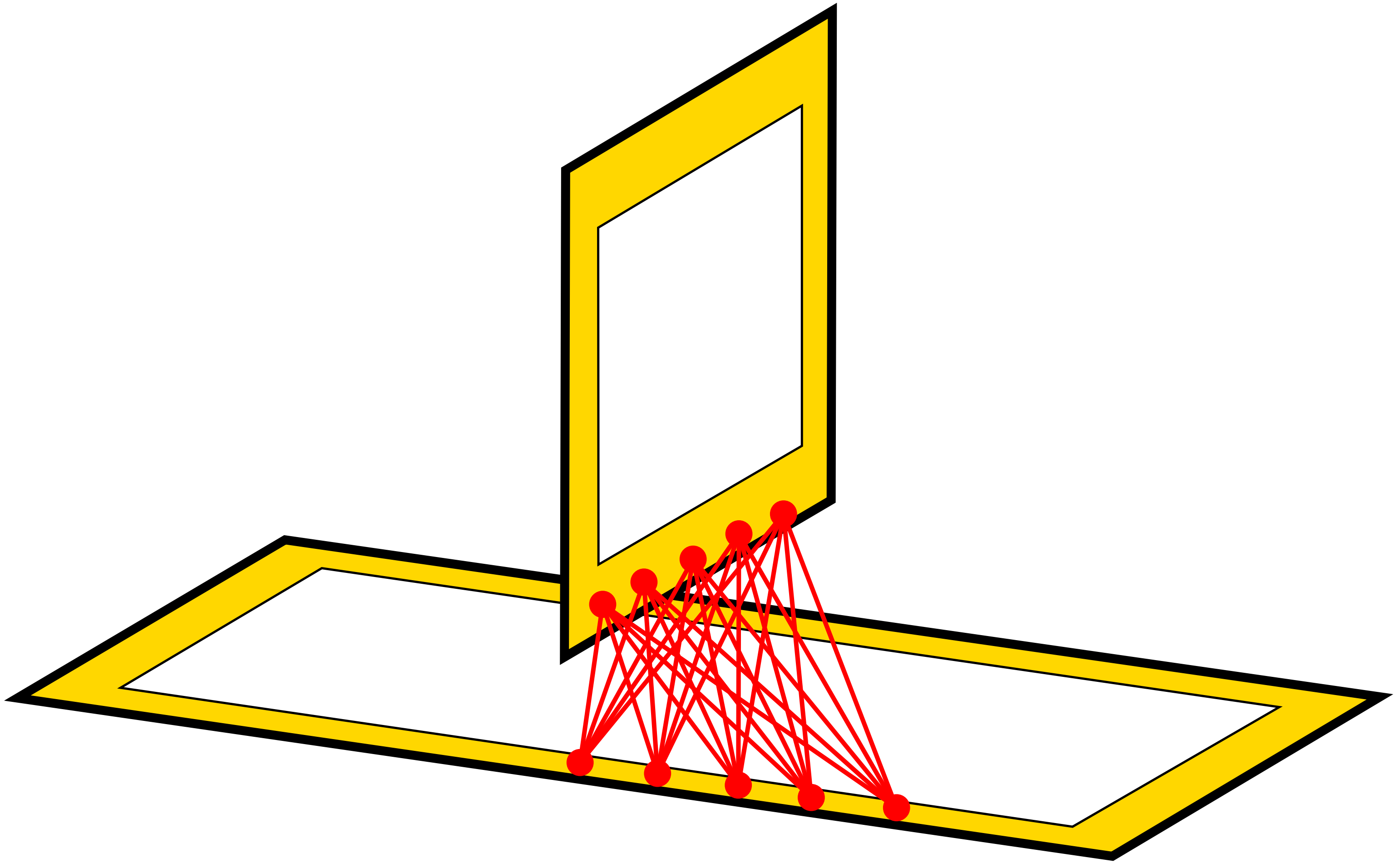
Regular - Regular

- Number of Delaunay edges in the regular zone : $O(n)$



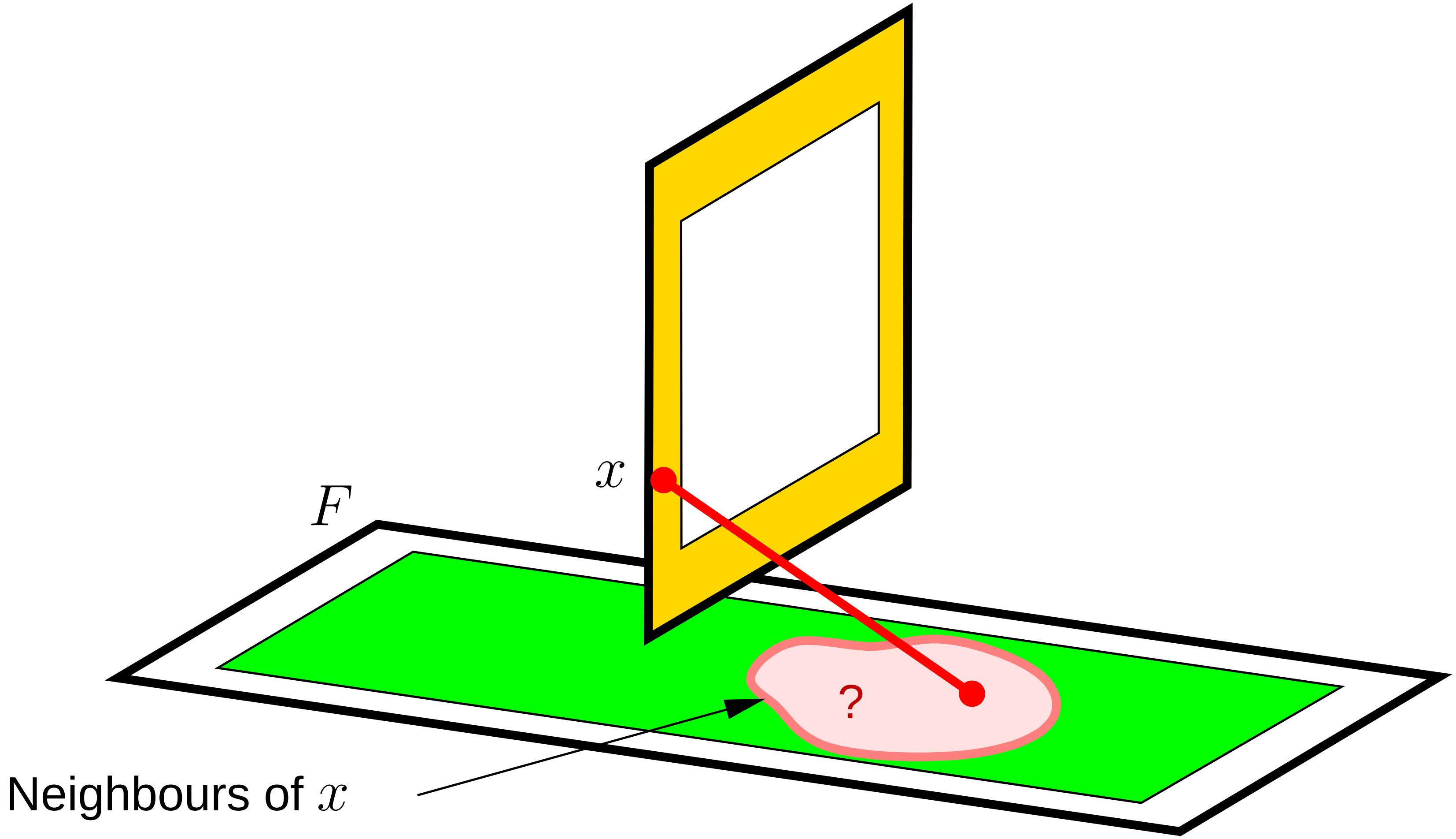
Singular - Singular

- Brutal force : $O(\sqrt{n}) \times O(\sqrt{n}) = O(n)$



Singular - Regular

- Locate the neighbours of x in F



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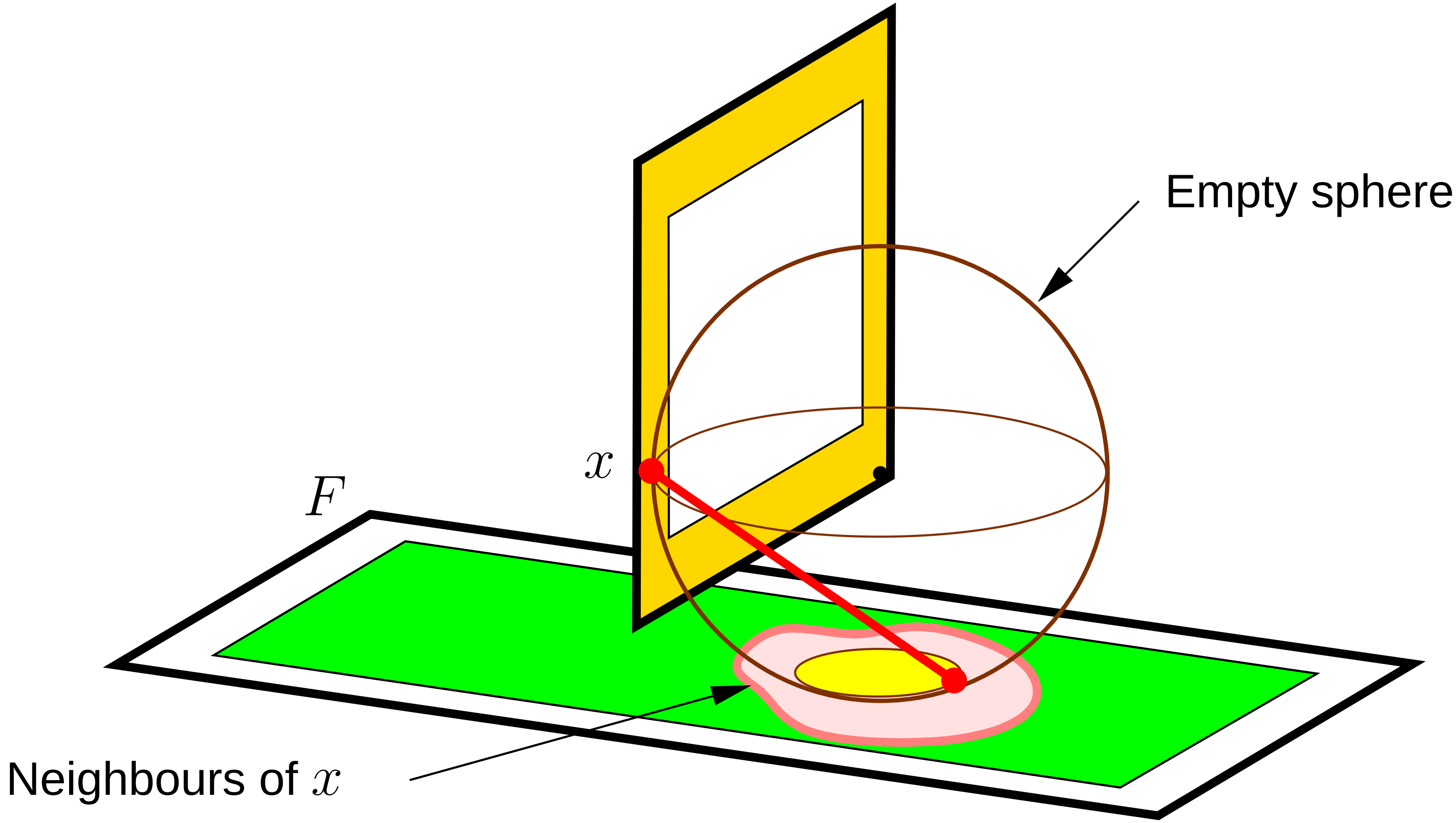

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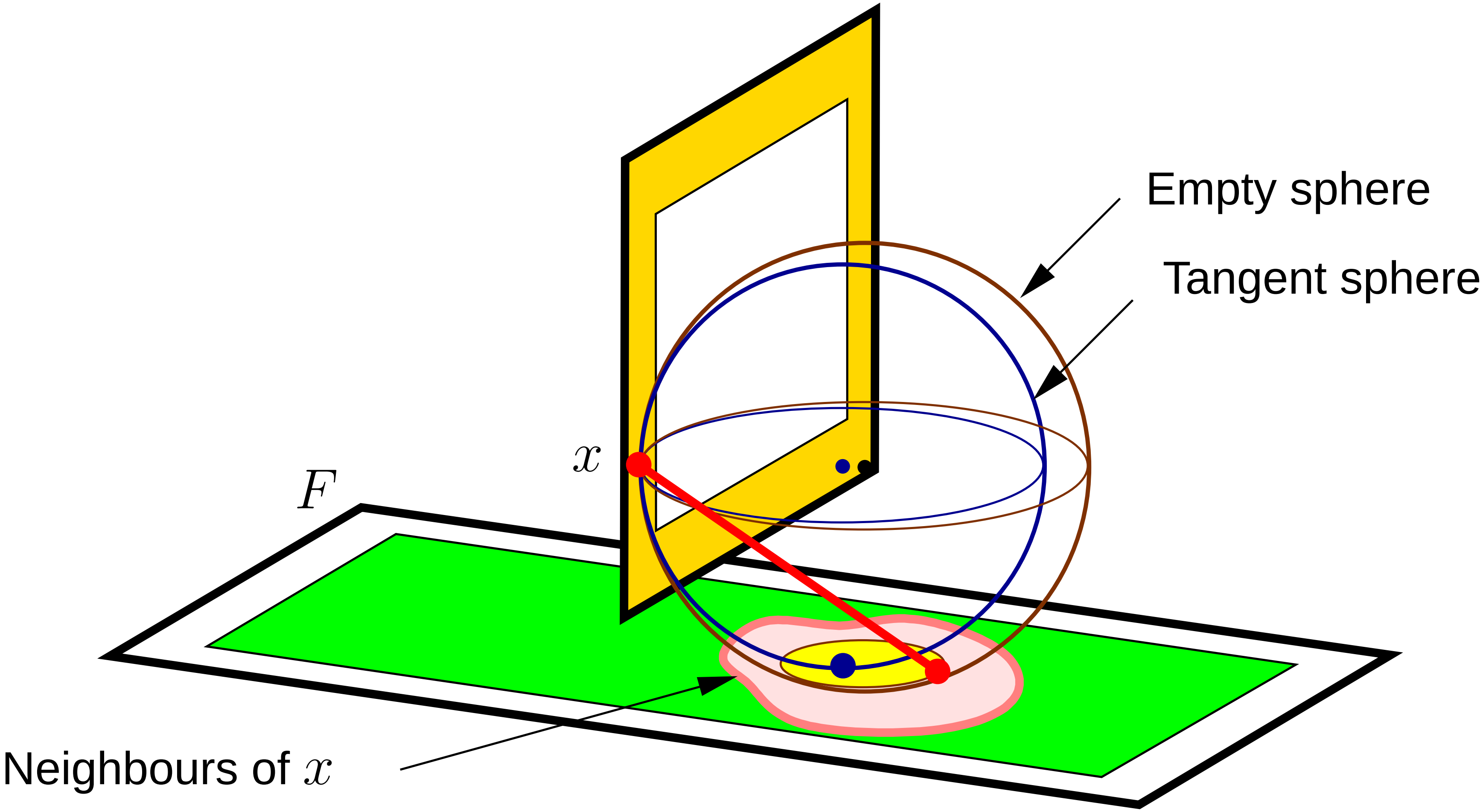
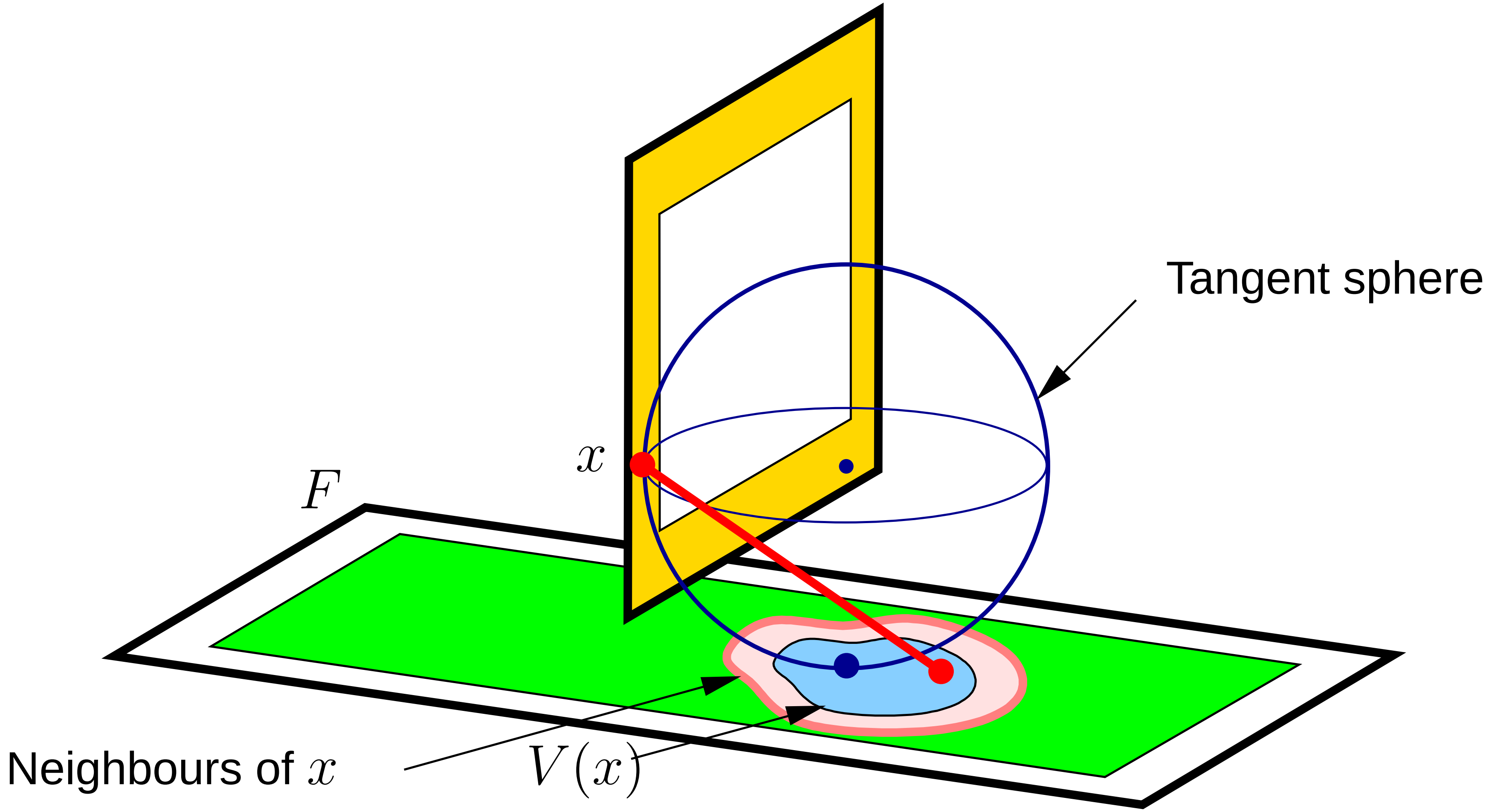
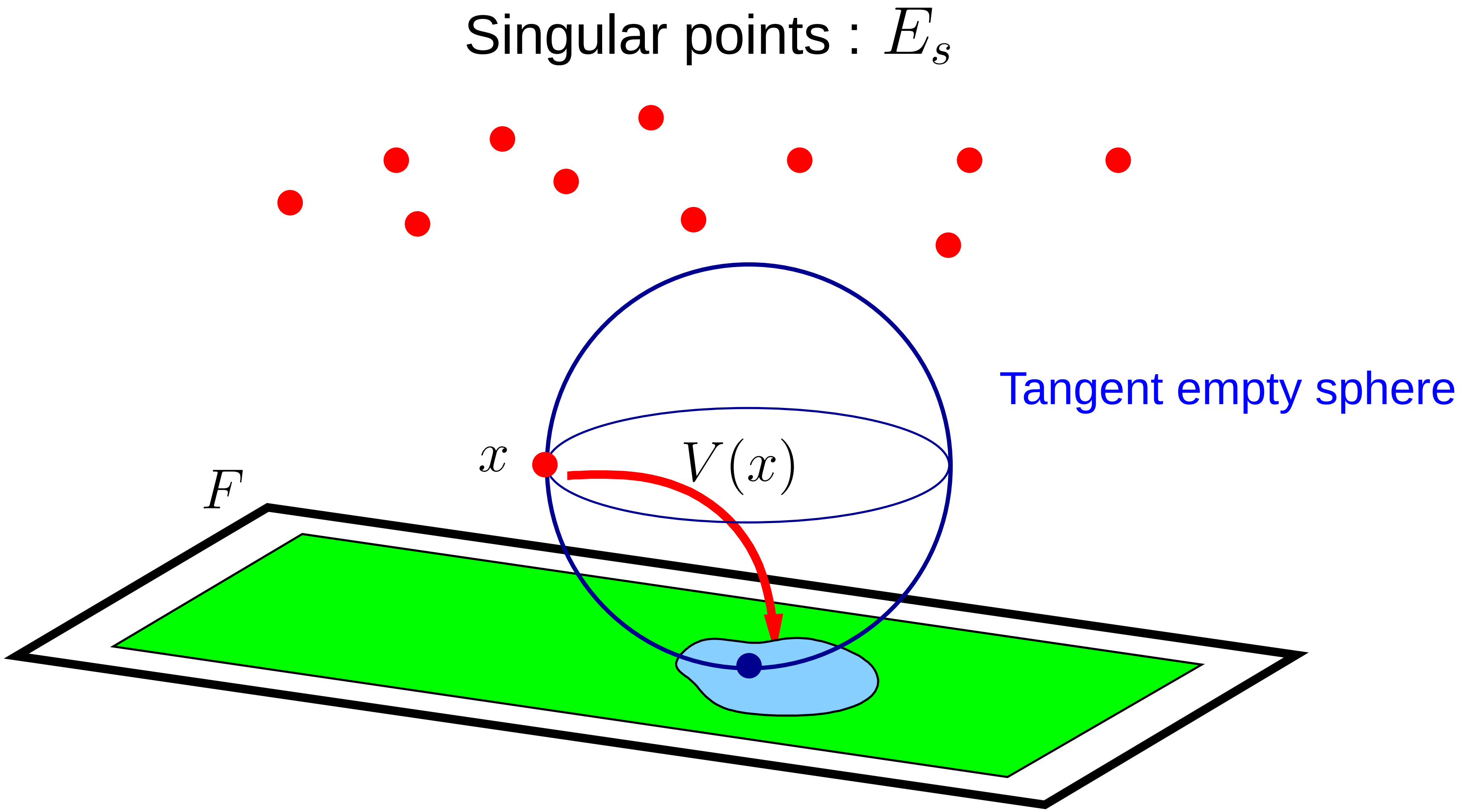


Singular - Regular

- Neighbours of x : $V(x)$ enlarged by 2ε



Singular - Regular



Singular - Regular

Singular points : E_s

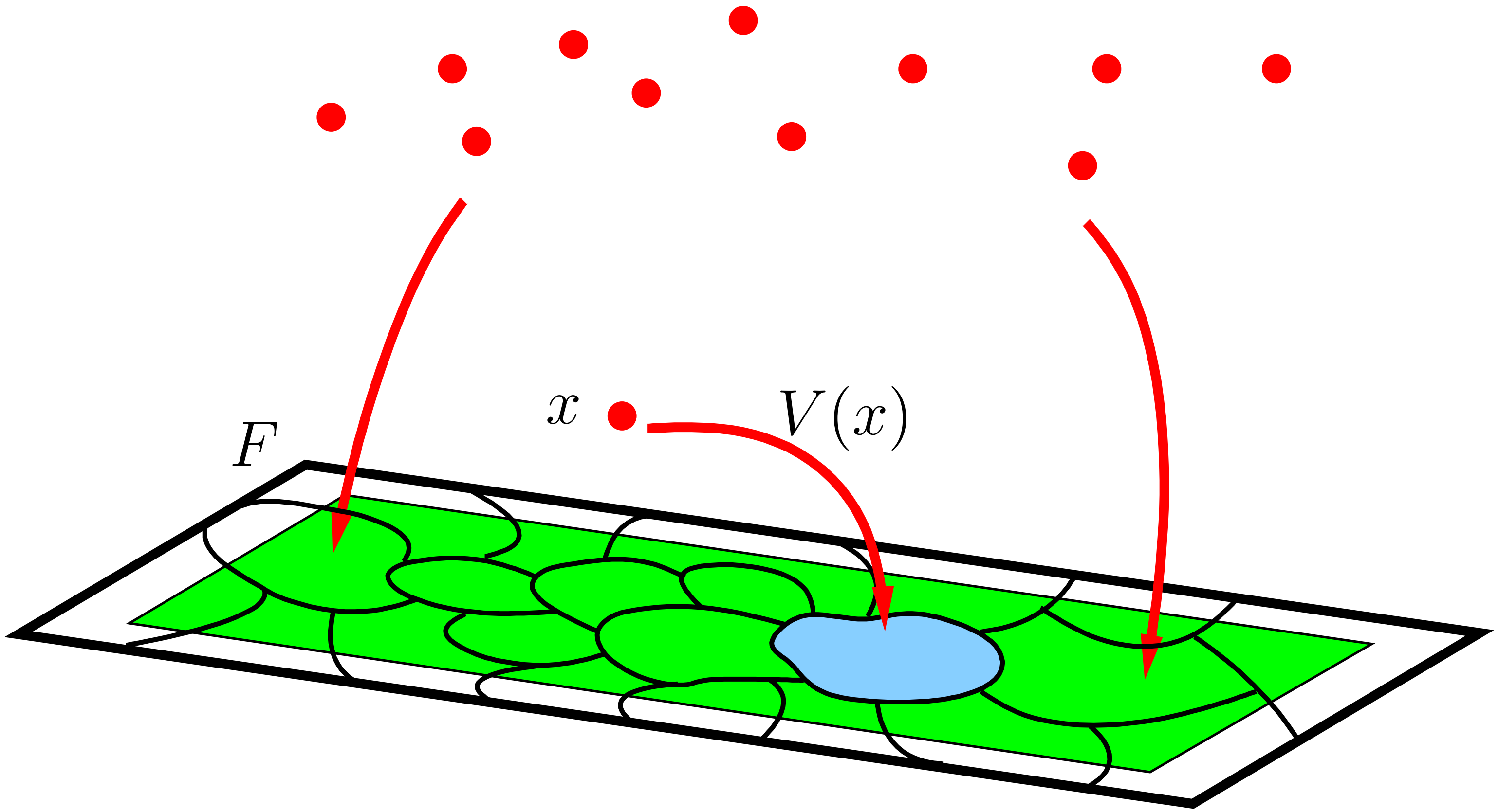


Diagram associated to F and points E_s

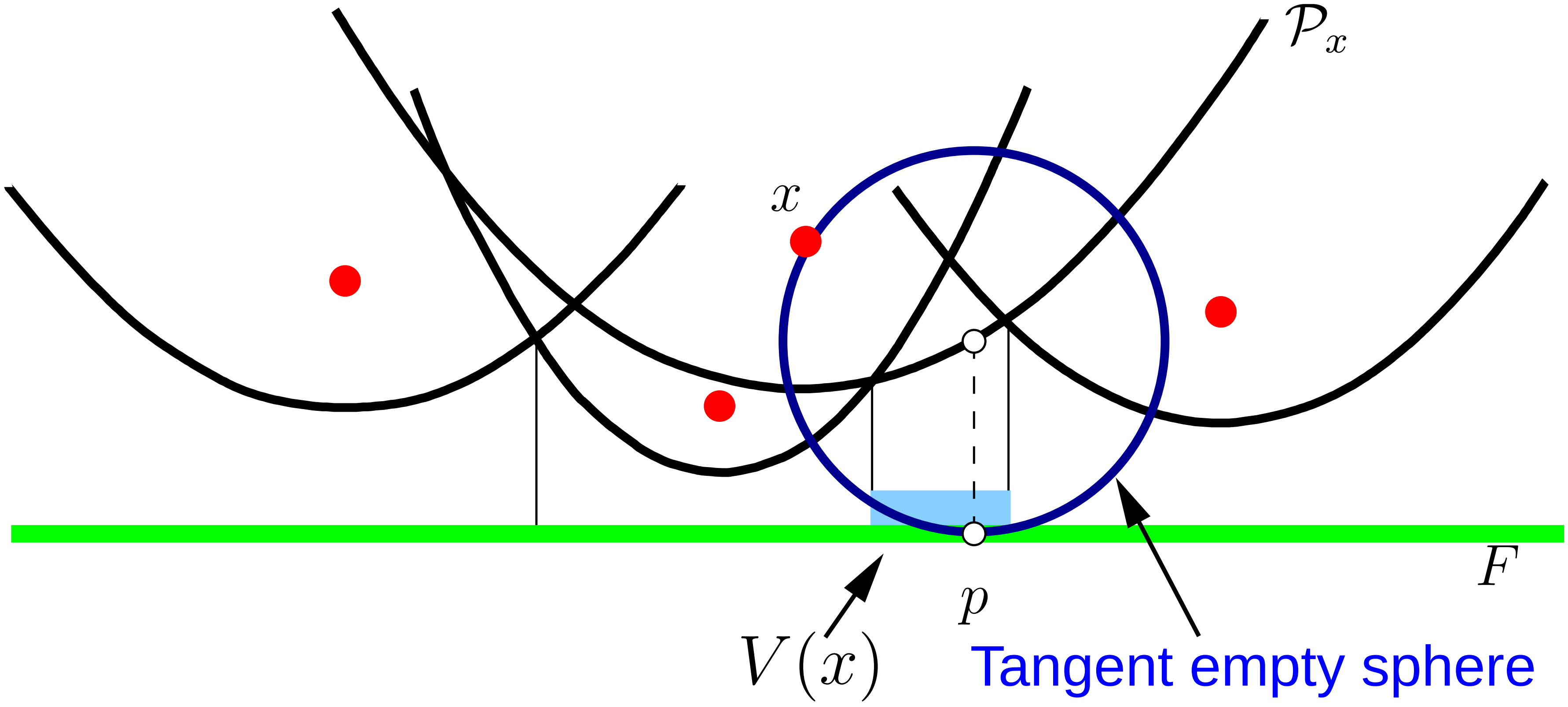
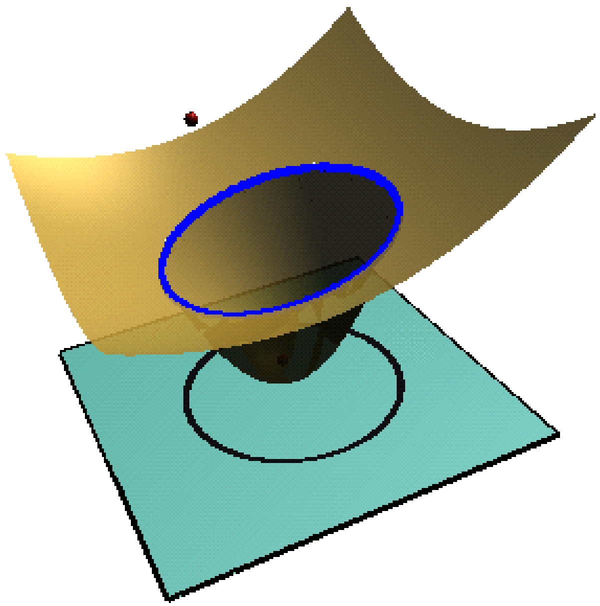
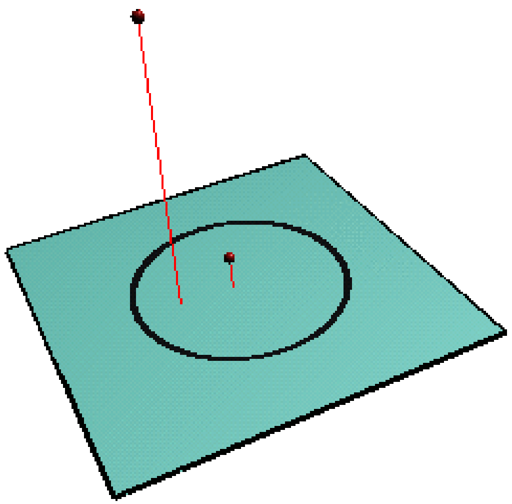


Diagram associated to F and points E_s

- Bissector of two points : a circle or a line





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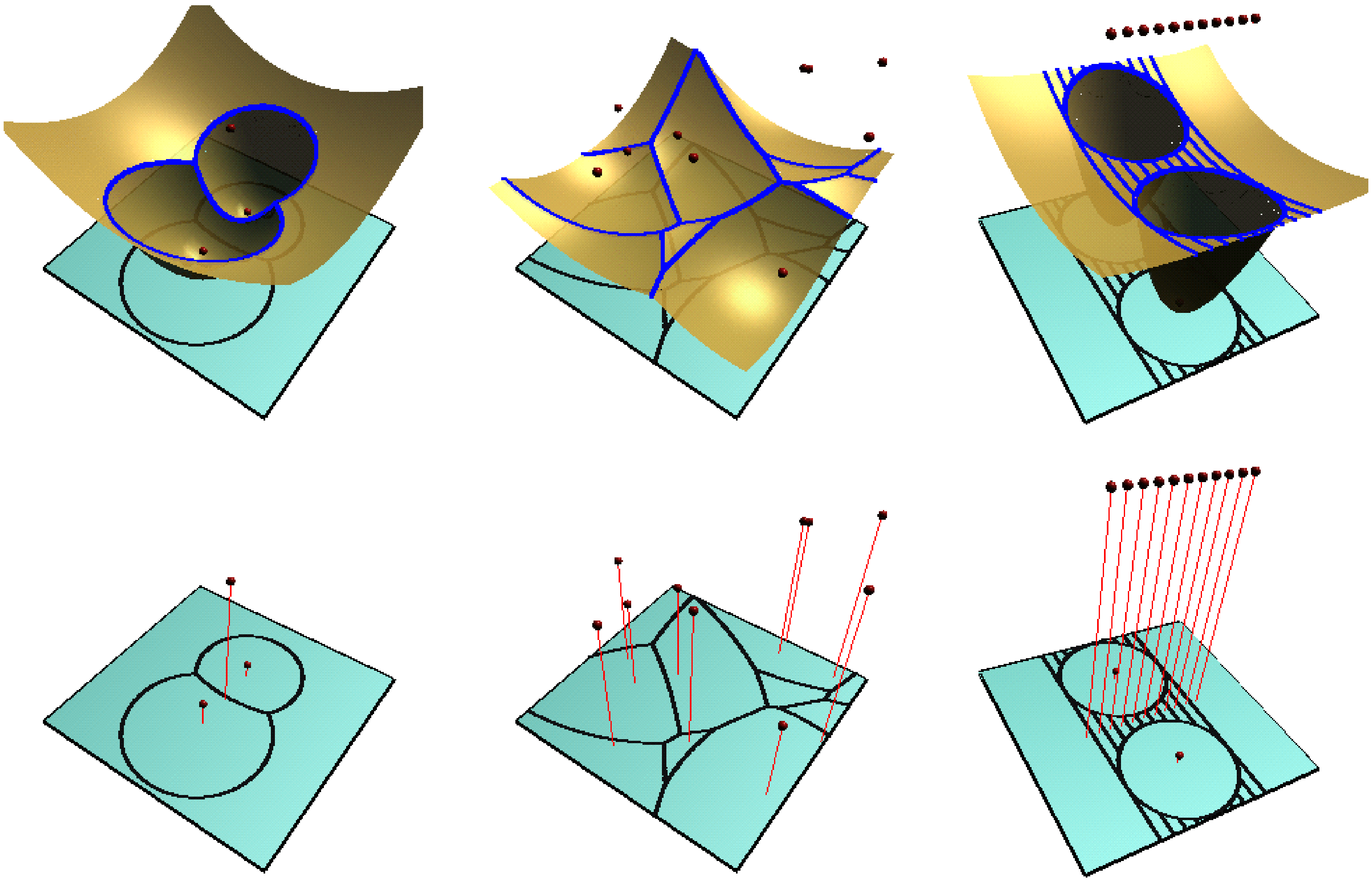
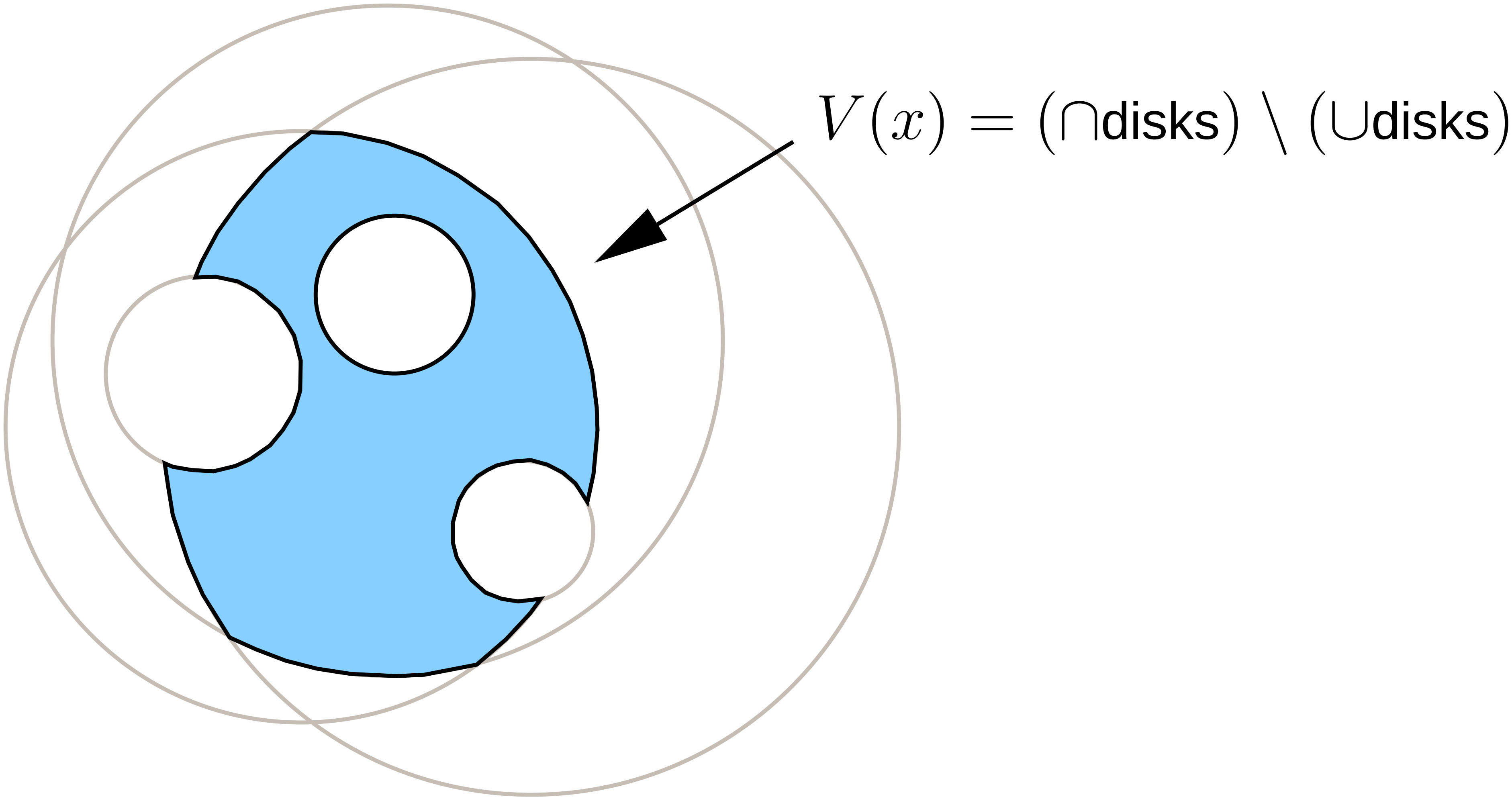


Delaunay edges between F and E_s



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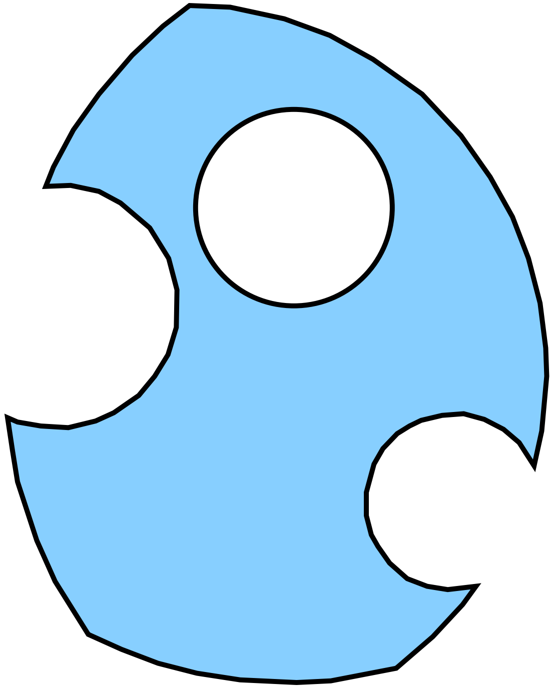
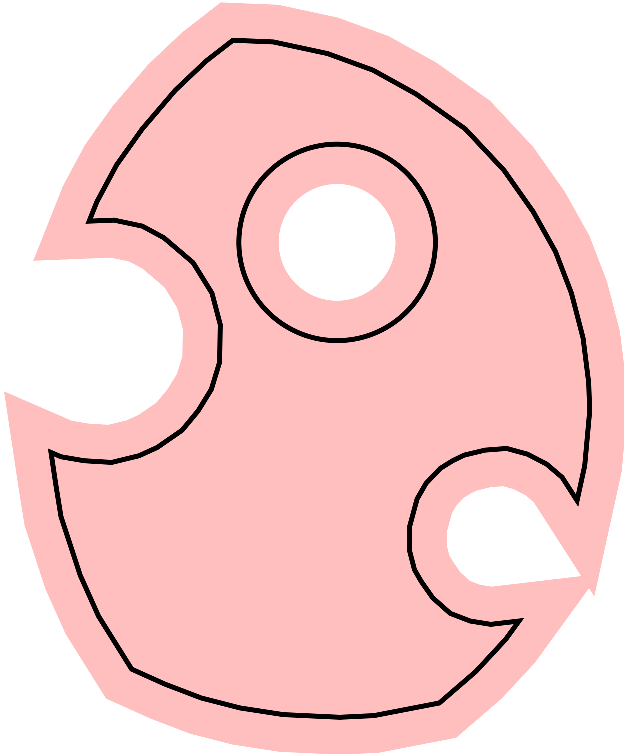
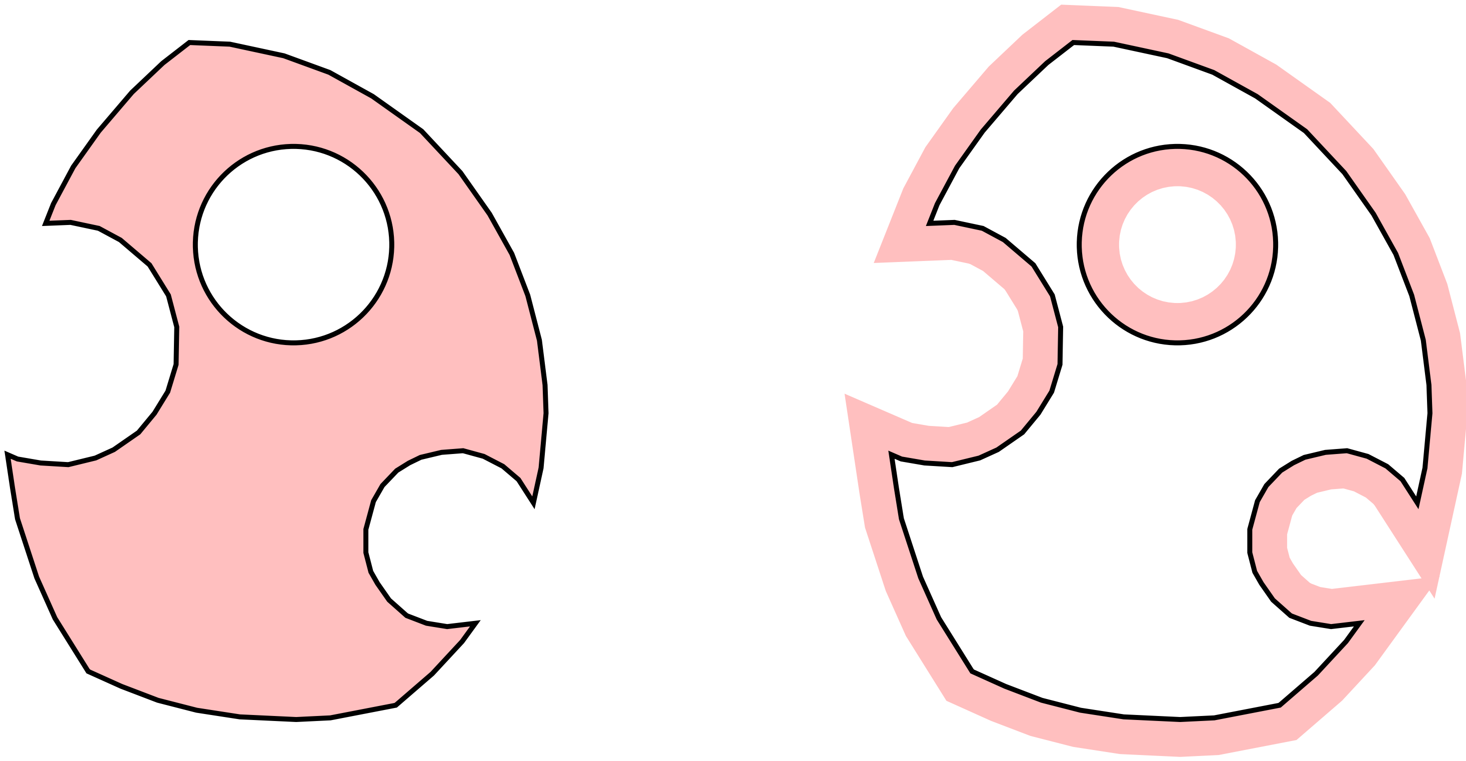


$V(x)$



Neighbours of x

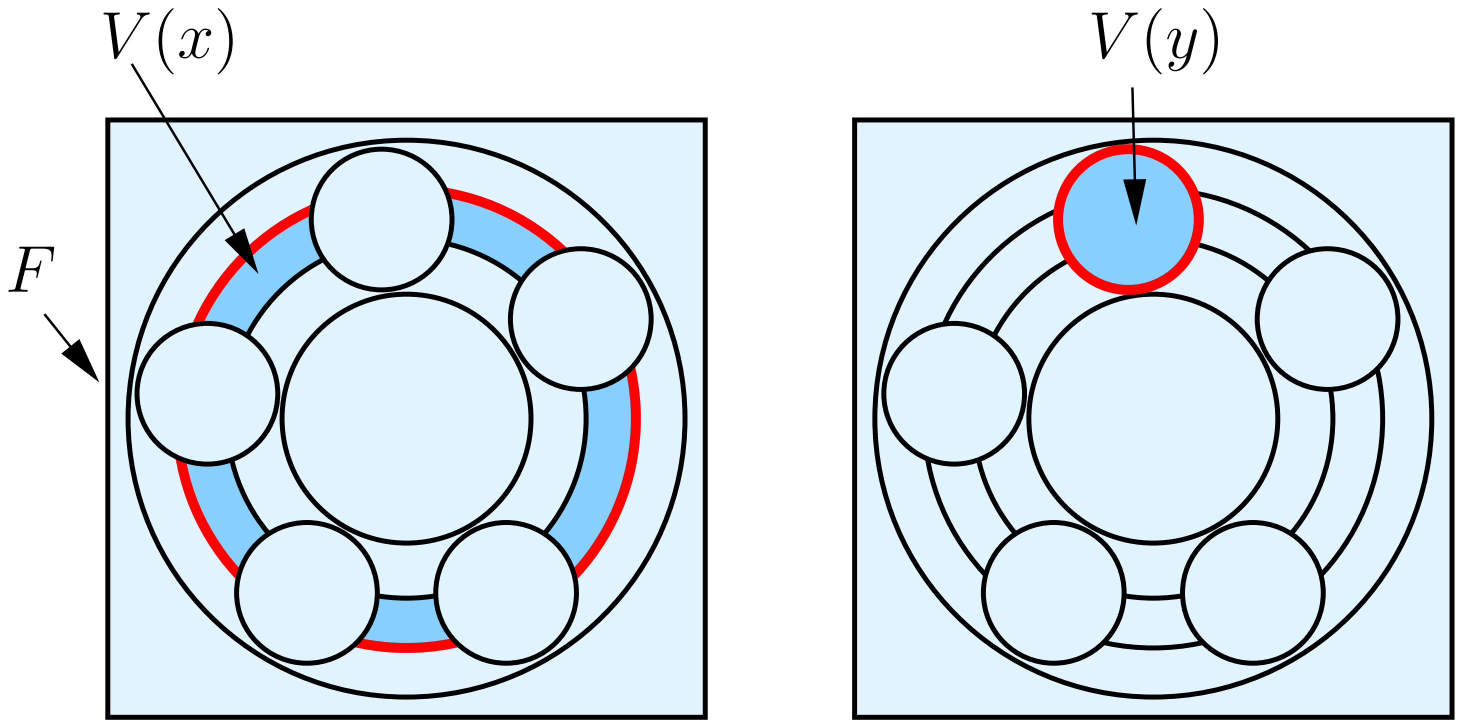
Delaunay edges between F and E_s



$$n(V(x)) \quad + \quad \text{length}(\partial V(x)) \times \sqrt{n}$$

Singular - Regular

- Length of edges $\leq n(E_s) \times \partial F = O(\sqrt{n})$



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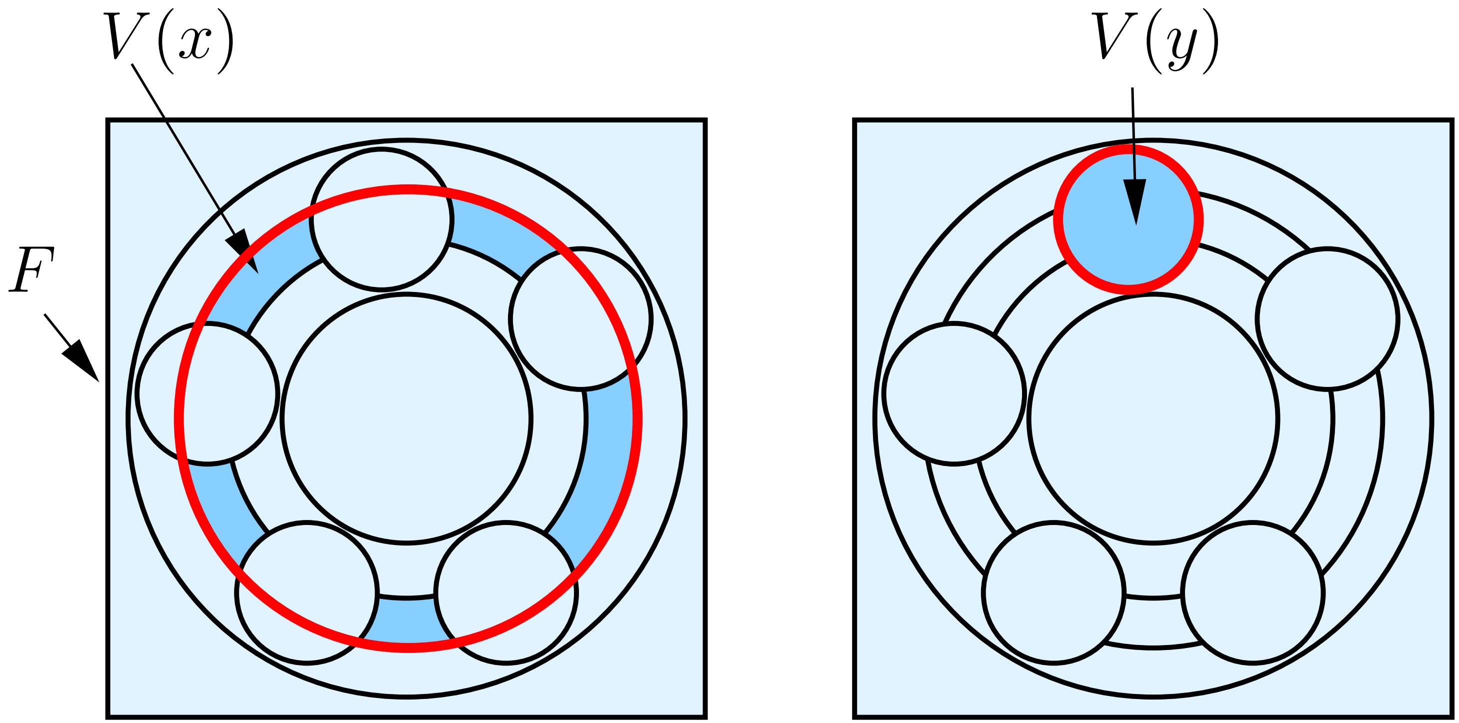

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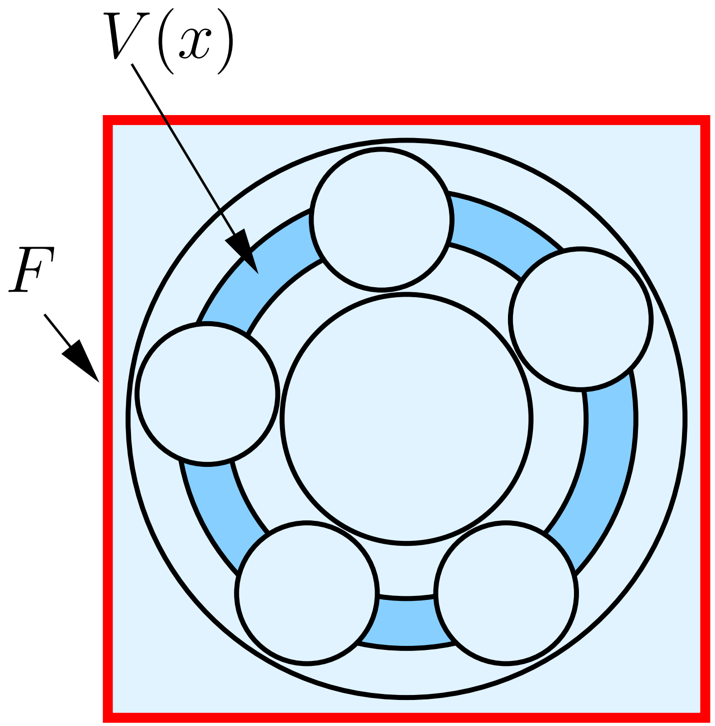

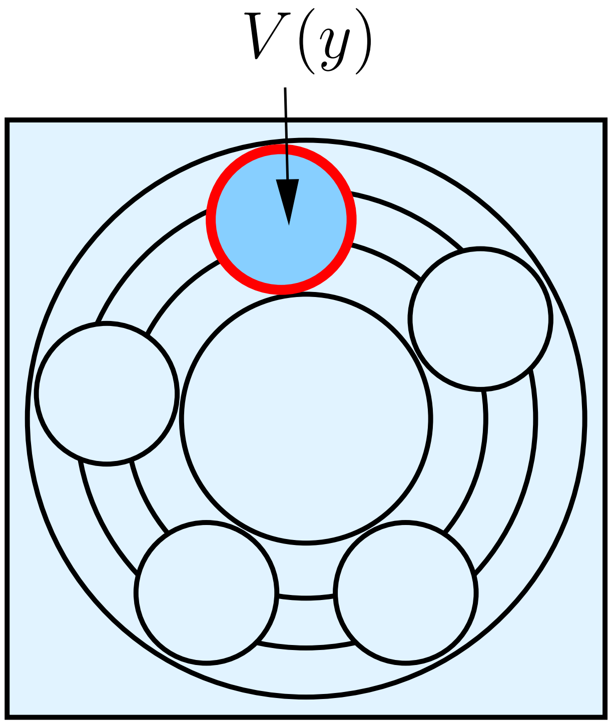

Main result

Let S be a polyhedral surface and E a (ε, κ)-sample of S of size $|E| = n$. The number of edges in the Delaunay triangulation of E is at most :

$$\left(1 + \frac{C\,\kappa}{2} + 612\,\pi\,\kappa^2\,\frac{L^2}{A}\right) n$$

C : number of facets

A : area

L : $\sum$ length(∂facet)

Conclusion and perspective

- Linear bound for polyhedral surfaces
- Extend this result to generic surfaces