A Tight Bound for the Delaunay Triangulation of Points on a Polyhedron

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Abstract

We show that the Delaunay triangulation of a set of n points distributed nearly uniformly on a p-dimensional polyhedron (not necessarily convex) in d-dimensional Euclidean space is $O(n^{\frac{d-k+1}{p}})$, where $k = \lceil \frac{d+1}{p+1} \rceil$. This bound is tight, and improves on the prior upper bound for most values of p.

1 Introduction

Overview. The Delaunay triangulation of a set of points is a fundamental geometric data structure, used, in low dimensions, in surface reconstruction, mesh generation, molecular modeling, geographic information systems, and many other areas of science and engineering. In higher dimensions, it is well-known [9] that the complexity of the Delaunay triangulation of n points is $O(n^{\lceil \frac{d}{2} \rceil})$ and that this bound is achieved by distributions of points along one-dimensional curves such as the moment curve. But points distributed uniformly in $\mathbb{R}^d$, for instance inside a d-dimensional ball, have Delaunay triangulations of complexity $O(n)$; the constant factor is exponential in the dimension, but the dependence on the number of points is linear. In an earlier paper [1], we began to fill in the picture in between these two extremes, that is, when the points are distributed on a manifold of dimension $2 \le p \le d-1$. We began with the easy case of a *p-dimensional polyhedron* P, and showed that for a particular (probably overly restrictive) sampling model the size of the Delaunay triangulation is $O(n^{(d-1)/p})$.

Main result. Here as in [1], we consider a fixed *p-dimensional polyhedron* P in d-dimensional Euclidean space $\mathbb{R}^d$. Our point set S is a *sparse ε-sample* from P. Sparse ε-sampling requires the sampling to be neither too sparse nor too dense. Let n be the number of points in S. We consider the complexity of the Delaunay triangulation of S, as $n \to \infty$, while P remains fixed. The main result in this paper is that the number of simplices of all dimensions is $O(n^{\frac{d-k+1}{p}})$ where $k = \lceil \frac{d+1}{p+1} \rceil$. The hidden constant factor depends, among other things, on the geometry of P, which is constant since P is fixed.

At the coarsest level, the idea of this proof is the same as that of [1]: we map Delaunay simplices to the medial axis and then use a packing argument to count them. The key new idea is the observation that when $k = \lceil \frac{d+1}{p+1} \rceil > 2$, the vertices of any Delaunay simplex, which must span $\mathbb{R}^d$, have to be drawn from more than two faces of P. This allows us to map Delaunay simplices to only the lower-dimensional submanifolds of the medial axis, induced by k or more faces. To realize this scheme, we introduce a new geometric structure, the *quasi medial axis*, which replaces the centers of tangent balls defining the medial axis with the centers of tangent annuli. In this paper, we only present an outline of the proof. Full details can be found in [2].

Prior work. The complexity of the Delaunay triangulation of a set of points on a two-manifold in $\mathbb{R}^3$ has received considerable recent attention, since such point sets arise in practice, and their Delaunay triangulations are found nearly always to have linear size. Golin and Na [6] proved that the Delaunay triangulation of a large enough set of points distributed uniformly at random on the surface of a fixed convex polytope in $\mathbb{R}^3$ has expected size $O(n)$. They later [7] established an $O(n \log^6 n)$ upper bound with high probability for the case in which the points are distributed uniformly at random on the surface of a non-convex polyhedron.

Attali and Boissonnat considered the problem using a sparse ε-sampling model similar to the one we use here, rather than a random distribution. For such a set of points distributed on a polygonal surface P, they showed that the size of the Delaunay triangulation is $O(n)$ [3]. In a subsequent paper with Lieutier [4] they considered "generic" smooth surfaces, and got an upper bound of $O(n \log n)$. Specifically, a "generic" surface is one for which each medial ball touches the surface in at most a constant number of points.

The genericity assumption is important. Erickson considered more general point distributions, which he characterized by the *spread*: the ratio of the largest inter-point distance to the smallest. The spread of a sparse ε-sample of n points from a two-dimensional manifold is $O(\sqrt{n})$. Erickson proved that the De-

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launay triangulation of a set of points in $\mathbb{R}^3$ with spread Δ is $O(\Delta^3)$. Perhaps even more interestingly, he showed that this bound is tight for $\Delta = \sqrt{n}$, by giving an example of a sparse ε-sample of points from a cylinder that has a Delaunay triangulation of size $\Omega(n^{3/2})$ [5]. Note that this surface is not generic and has a degenerate medial axis.

To the best of our knowledge, ours [1] is the only prior result for $d > 3$.

2 Statement of Theorem

In this section, we introduce the setting for our result. Given a polyhedron $P \subseteq \mathbb{R}^d$ and a point x on P, let F_x be the unique face that contains x. We say that a set of points $S \subseteq P$ is a *λ-sparse ε-sample* of P iff it satisfies the following two conditions:

`Density:` Every point x in P is at distance ε or less to a point s in S lying on the closure of F_x.

`Sparsity:` Every closed d-ball with radius $6d\varepsilon$ contains at most λ points of S.

Note that our density condition implies that all faces of all dimensions are uniformely sampled, not just faces with highest dimension as in [3, 4]. Hereafter, we consider λ to be a constant. The number n of points in a λ-sparse ε-sample of a p-dimensional polyhedron is related to ε by $n = \Theta(\varepsilon^{-p})$. Thus, as n tends to infinity, ε tends to zero. We are now ready to state our main result:

Theorem 1 *Let S be a λ-sparse ε-sample of a p-dimensional polyhedron P in $\mathbb{R}^d$, and let n be the number of points in S. The Delaunay triangulation of S has size $O(n^{\frac{d-k+1}{p}})$ where $k = \lceil \frac{d+1}{p+1} \rceil$.*

Note that our result requires no non-degeneracy assumption, neither on P nor on S.

3 Essential quasi medial axes

In this section, we introduce the *ε-quasi k-medial axis*, a variant of the medial axis based on tangent annuli rather than tangent balls, which is the key geometric object in our proof. We then define the part of the ε-quasi k-medial axis to which Delaunay simplices will be mapped: the essential ε-quasi k-medial axis (considering only the parts of dimension at most $d-k+1$ and lopping off the parts which extends to infinity). Along the way, we give a tool to identify lower-dimensional parts of the ε-quasi k-medial axis.

3.1 Quasi medial axes

We start by defining ε-quasi k-medial axes. We say that a $(d-1)$-sphere Σ is *tangent* to a face F at point x if both the closure of F and the affine space spanned by F intersect Σ in a unique point x. An annulus with center z, inner radius r and outer radius R is the set of points x whose distance to the center satisfies $r \leq \|x-z\| \leq R$. The boundary of an annulus consists of two $(d-1)$-spheres and we call the smallest one the *inner sphere* and the largest one the *outer sphere*. Extending what we just defined for spheres, we say that an annulus A is *tangent* to F at x if one of the two spheres bounding A is tangent to F at x. Point x is called a *tangency point* of A. An annulus is *P-empty* if its inner sphere bounds a d-ball whose interior does not intersect P. An annulus is called *ε-thin* if the difference between the outer and inner radii squared is $R^2 - r^2 = \varepsilon^2$.

Definition 1 *The ε-quasi k-medial axis $\mathcal{M}^k(P,\varepsilon)$ of P is the set of points $z \in \mathbb{R}^d$ for which for the largest P-empty ε-thin annulus centered at z, $A(z,\varepsilon)$, is tangent to at least k faces of P (see Figure 1).*

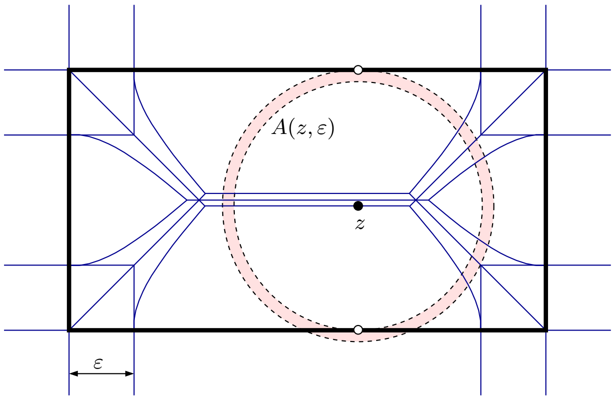

Figure 1: A rectangle and its ε-quasi 2-medial axis composed of 16 half-lines, 5 segments and 8 pieces of hyperbolas.

3.2 Identifying lower-dimensional strata

Because P might be degenerate, we must introduce a tool to identify the parts of $\mathcal{M}^k(P,\varepsilon)$ which have dimension $d-k+1$ or less very carefully. We recall that a stratification of a subset $X \subseteq \mathbb{R}^d$ is a filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X_j = X$$

by subspaces such that the set difference $X_i \setminus X_{i-1}$ is a i-dimensional manifold, called the i-dimensional *stratum* of X. In particular, semi-algebraic sets admit a stratification [8] and since ε-quasi k-medial axes of polyhedra are piecewise semi-algebraic, they also admit a stratification.

Definition 2 *We say that k faces $F_1, \ldots, F_k$ are in*dependent *if none of them is contained in the affine* *space spanned by the union of the others, that is for* $1 \leq i \leq k$,

$$F_i \quad \not\subseteq \quad \mathrm{Aff}(F_1 \cup \cdots \cup \widehat{F_i} \cup \cdots \cup F_k),$$

where the symbol $\widehat{}$ *over* F_i *indicates that it is omitted in the union.*

Lemma 2 *Let* $z \in \mathcal{M}^k(P, \varepsilon)$ *and suppose that* $A(z, \varepsilon)$ *is tangent to j faces amongst which k faces are independent. Then, z lies on a stratum of* $\mathcal{M}^k(P, \varepsilon)$ *of dimension* $d - k + 1$ *or less.*

3.3 Essential part

The ε-quasi k-medial axis in general extends to infinity and therefore can have an infinite volume. In this section, we select a subset of the ε-quasi k-medial axis called the essential ε-quasi k-medial axis, $\bar{\mathcal{M}}^k(P, \varepsilon)$ in such a way that all its strata will have a finite volume bounded by a constant that does not depend on ε. For this, we need some definitions. We say that a hyperplane *supports* $X \subseteq \mathbb{R}^d$ if it has non-empty intersection with the boundary of X and empty intersection with the interior of X.

Definition 3 *A point z is* ε-essential *if there exists no hyperplane supporting the convex hull of P and containing all faces tangent to* $A(z, \varepsilon)$.

It follows immediately from the definition that:

Lemma 3 *If the union of faces tangent to* $A(z, \varepsilon)$ *spans* $\mathbb{R}^d$*, then z is* ε*-essential.*

Definition 4 *The* essential ε-quasi k-medial axis, $\bar{\mathcal{M}}^k(P, \varepsilon)$*, is the set of* ε*-essential points lying on the union of the i-dimensional strata of the* ε*-quasi k-medial axis over all* $i \leq d - k + 1$.

Lemma 4 *For* ε *smaller than the diameter of P, the i-dimensional stratum of the* ε*-quasi k-medial axis has a i-dimensional volume bounded by a constant, that does not depend on* ε.

4 Covering Delaunay spheres

The goal of this section is to prove that the intersection of a p-dimensional polyhedron P with any Delaunay sphere Σ is contained in the cover of some point z on the essential ε-quasi k-medial axis, for $k = \lceil \frac{d+1}{p+1} \rceil$. We first state crucial properties of Delaunay spheres and polyhedra before defining the cover of a point. The first property is induced by our sampling condition.

Definition 5 *We say that a sphere* Σ *with center z is* ε-almost P-empty *if* $\Sigma \subseteq A(z, \varepsilon)$.

Lemma 5 *Delaunay spheres are* ε*-almost P-empty.*

The second property concerns polyhedra.

Definition 6 *We say that a polyhedron P is* k-reductible *if for any collection of* $k-1$ *faces* $\{F_1, \ldots, F_{k-1}\}$ *of P, there exists a hyperplane that contains the union* $\bigcup_{i=1}^{k-1} F_i$.

Lemma 6 *Any p-dimensional polyhedron of* $\mathbb{R}^d$ *is* $\lceil \frac{d+1}{p+1} \rceil$*-reductible.*

We now define the cover of a point $z \in \mathbb{R}^d$. Writing $\pi_x(z)$ for the orthogonal projection of z onto the tangent plane of $x \in P$, we say that x is a critical point of the distance-to-z function if $\pi_x(z) = x$. We define $\chi(z, \varepsilon)$ as the set of critical points lying in $P \cap A(z, \varepsilon)$ and the cover of z as:

$$\mathrm{Cover}(z, \varepsilon) = \bigcup_{x \in \chi(z, \varepsilon)} B(x, 5d\varepsilon).$$

Lemma 7 *Consider a k-reductible polyhedron P that spans* $\mathbb{R}^d$*. For every* ε*-almost P-empty sphere* Σ*, there exists a point* $z \in \bar{\mathcal{M}}^k(P, \varepsilon)$ *such that*

$$\Sigma \cap P \subseteq \mathrm{Cover}(z, \varepsilon).$$

In the next section, it will be convenient to use a slightly different notion of cover. Let $\Pi(z)$ be the set of orthogonal projections of z onto the planes supporting faces of P. We define the extended cover of point z as

$$\mathrm{ExtendedCover}(z, \varepsilon) = \bigcup_{x \in \Pi(z)} B(x, 6d\varepsilon).$$

Lemma 8 *For every points z and z′ with* $\|z - z'\| \leq \varepsilon$:

$$\mathrm{Cover}(z, \varepsilon) \subseteq \mathrm{ExtendedCover}(z', \varepsilon).$$

5 Size of Delaunay triangulation

In this section, we collect results from previous sections and establish our upper bound on the number of Delaunay simplices. We then prove that our bound is tight. We recall that the number of points in a λ-sparse ε-sample S of a p-dimensional polyhedron P is $n = \Theta(\epsilon^{-p})$ and that the i-faces of P have $\Theta(\varepsilon^{-i})$ points of S [1].

5.1 Upper bound

Without loss of generality, we may assume that the polyhedron P spans $\mathbb{R}^d$. An ε-sample of the essential ε-quasi k-medial axis is a subset $M \subseteq \bar{\mathcal{M}}^k(P, \varepsilon)$ such that every point $x \in \bar{\mathcal{M}}^k(P, \varepsilon)$ is at distance no more than ε to a point $z \in M$, $\|x - z\| \leq \varepsilon$. We claim that we can construct an ε-sample M of $\bar{\mathcal{M}}^k(P, \varepsilon)$ in such a way that the i-dimensional stratum of the essential ε-quasi k-medial axis receives $O(\varepsilon^{-i})$ points of M and the number of points in M is $m = O(\varepsilon^{-(d-k+1)})$. This is a consequence of Lemma 4 which says that the i-dimensional volume of the i-dimensional stratum of $\bar{\mathcal{M}}^k(P, \varepsilon)$ is bounded by a constant that does not depend on ε. To establish our upper bound, we map each Delaunay simplex $\sigma \in \mathrm{Del}(S)$ to a point $z \in M$. Consider a Delaunay sphere Σ passing through the vertices of σ. By Lemma 5, Delaunay spheres are ε-almost P-empty. We can therefore combine Lemma 6, Lemma 7 and Lemma 8 and get that for $d \geq 2$ and $k = \lceil \frac{d+1}{p+1} \rceil$, there exists a point $z \in M$ such that

$$\Sigma \cap P \subseteq \mathrm{ExtendedCover}(z, \varepsilon)$$

The extended cover of z is a union of d-balls of radius $6d\varepsilon$, one for each face of the polyhedron and therefore, it contains a constant number of points of S. It follows that the number of simplices that we can form by picking points in the extended cover of z is constant. Hence, each point $z \in M$ is charged with a constant number of Delaunay simplices and using $n = \Omega(\varepsilon^{-p})$, we get that the number of Delaunay simplices is

$$O(m) = O(\varepsilon^{-(d-k+1)}) = O(n^{\frac{d-k+1}{p}}),$$

where $k = \lceil \frac{d+1}{p+1} \rceil$.

5.2 The bound is tight

We now prove that our upper bound is tight. Consider a set of $d+1$ affinely independent points that we partition into $k = \lceil \frac{d+1}{p+1} \rceil$ groups $Q_1, \ldots, Q_k$ in such a way that the maximum number of points in Q_i, over all $i \in [1, k]$, is $p + 1$. Writing q_i for the dimension of the affine space spanned by Q_i, we have

$$\sum_{i=1}^{k} q_i = d - k + 1. \quad (1)$$

Letting C_i be the convex hull of Q_i, we consider the polyhedron $P = \bigcup_{i=1}^{k} C_i$. Let S be a λ-sparse ε-sample of P. The simplex $\sigma = \{s_1, \ldots, s_k\}$ obtained by picking a sample point $s_i \in S \cap C_i$ for $1 \leq i \leq k$ belongs to the Delaunay triangulation. Indeed, since the points $s_1, \ldots, s_k$ are affinely independent, there exists a $(d-1)$-sphere Σ tangent to P at s_i for $1 \leq i \leq k$, whose center lies on the 0-quasi k-medial axis of P. By construction, this sphere encloses no sample point of S in its interior, showing that σ is a Delaunay simplex. Since C_i contains $\Omega(\varepsilon^{-q_i})$ points of S, the amount of Delaunay simplices that we can construct this way is at least

$$\Omega(\varepsilon^{-q_1} \times \cdots \times \varepsilon^{-q_k}) = \Omega(\varepsilon^{-(d-k+1)}) = \Omega(n^{\frac{d-k+1}{p}}).$$

6 Conclusion

This paper answers only the first of many possible questions about the complexity of the Delaunay triangulations of points distributed nearly uniformly on manifolds. Similar bounds for smooth surfaces rather than polyhedra would be of more practical interest. The proof in this paper seems to rely on some properties specific to polyhedra, particularly that sample points on k faces are needed to form a simplex. On the other hand, the tight bound seems to be "right", at least in the sense that it agrees with the well-known bounds in the cases $p = 1$ and $p = d$.

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