Complexity of the Delaunay Triangulation of Points on Surfaces : the Smooth Case*

(Extended Abstract)

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ABSTRACT

It is well known that the complexity of the Delaunay triangulation of N points in $\mathbb{R}^3$, i.e. the number of its faces, can be $\Omega(N^2)$. The case of points distributed on a surface is of great practical importance in reverse engineering since most surface reconstruction algorithms first construct the Delaunay triangulation of a set of points measured on a surface.

In this paper, we bound the complexity of the Delaunay triangulation of points distributed on generic smooth surfaces of $\mathbb{R}^3$. Under a mild uniform sampling condition, we show that the complexity of the 3D Delaunay triangulation of the points is $O(N \log N)$.

Categories and Subject Descriptors

F.2.2 [**Theory of Computation**]: Analysis of Algorithms and Problem Complexity—*Geometrical problems and computations*; I.3.5 [**Computing Methodologies**]: Computer Graphics—*Computational Geometry and Object Modeling*

General Terms

Theory, Algorithms, Performance

Keywords

Delaunay triangulation, Voronoi diagram, complexity, generic surfaces, smooth surfaces, reconstruction.

1. INTRODUCTION

It is well known that the complexity of the Delaunay triangulation of N points in $\mathbb{R}^3$, i.e. the number of its faces, can be as large as $\Omega(N^2)$.

The case of points distributed on a surface is of great practical importance in reverse engineering since most surface reconstruction algorithms first construct the Delaunay triangulation of a set of points measured on a surface, see e.g. [1, 5, 8, 9]. The time complexity of those methods therefore crucially depends on the complexity of the triangulation of points scattered over a surface in $\mathbb{R}^3$. Moreover, since output-sensitive algorithms are known for computing Delaunay triangulations [6, 7], better bounds on the complexity of the Delaunay triangulation would immediately imply improved bounds on the time complexity of computing the Delaunay triangulation.

Although the question has been open for a long time [4], it is only recently that several results have been obtained. The results differ on the type of surface considered and also on the sampling assumptions. Erickson [10, 11] proved that the complexity of the Delaunay triangulation of any finite set of points is $O(\Delta^3)$ if Δ denotes the *spread*, i.e. the ratio between the largest and the smallest interpoint distances. If the points belong to a fixed and regularly sampled surface, $\Delta = \Theta(\sqrt{N})$ and the bound on the complexity of the Delaunay triangulation becomes $O(N\sqrt{N})$. It is to be observed that the above bound is only meaningful if no two points are too close, a quite restrictive sampling condition. Erickson proved also a $\Omega(N\sqrt{N})$ lower bound for points nicely distributed on a cylinder.

Golin and Na [12, 13] and Attali and Boissonnat [2] have considered the polyhedral case. Golin and Na assume that the sample points are chosen uniformly at random on the surface. They show that the expected complexity of the Delaunay triangulation is $O(N \log^4 N)$. Attali and Boissonnat consider so-called (ε, κ)-samples and prove a deterministic linear bound. An (ε, κ)-sample is a sufficiently dense sample that cannot be arbitrarily dense locally: more precisely, any ball of radius ε centered on the surface contains at least one and at most κ sample points.

In this paper, we consider the case of points distributed on a *smooth* surface. We prove that the complexity of the Delaunay triangulation of an (ε, κ)-sample of points scattered over a fixed *generic* surface is $O(N \log N)$. A surface

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is generic if, roughly, the ridges, i.e. the points on the surface where one of the principal curvature is locally maximal, is a finite set of curves whose total length is bounded. In particular, spheres and cylinders are excluded.

2. DEFINITIONS AND FIRST PROPERTIES

2.1 Notations

S designates a surface and $A \subset S$ a finite set of N sample points of S.

For $p, q \in \mathbb{R}^3$, $d(p,q)$ denotes the usual Euclidean distance while, if $p, q \in S$, $d_S(p,q)$ denotes the geodesic distance on S, i.e. the infimum of the lengths of the paths on S from p to q. If p and q belong to two distinct connected components of S, we let $d_S(p,q) = +\infty$.

In the sequel, we consider spaces such as $S \times \mathbb{R}^3$ and $S \times S \times \mathbb{R}^3$. They are considered as metric spaces : d and d_S induce a metric on these product spaces by taking the maximum of the distances between respective coordinates.

For $F \subset S$, let $F^{+\delta} = \{p \in S, d_S(p,F) < \delta\}$.

$\sigma_{(c,R)}$ (resp. $B_{(c,R)}$) denotes the sphere (resp. ball) of center c and radius R. If σ is a sphere, $B(\sigma)$ denotes the closed ball whose boundary is σ.

For $p \in S$, $D_S(p,R)$ denotes the geodesic disk of center p and radius R.

$\sharp(A)$ denotes the cardinal (*i.e.* number of elements) of A.

2.2 (ε,κ)-samples

Definition 1. A finite subset A of points of S is said to be an ε-sample of S iff, for any point $p \in S$, $D_S(p,\varepsilon)$ contains at least one point of A. If, in addition, $D_S(p,\varepsilon)$ contains at most κ points of A, A is called an (ε,κ)-sample.

In the rest of the paper, we assume that A is an (ε,κ)-sample of S. We first give an upper bound on the number of sample points in a region $R \subset S$.

LEMMA 1. *Let A be an (ε,κ)-sample of S. One can find a constant ϵ_0 such that for $\varepsilon \le \epsilon_0$ and for any $R \subset S$, we have:*

$$\sharp(R \cap A) \le \frac{8\kappa}{\pi} \times \frac{\text{area}\left(R^{+\frac{\varepsilon}{2}}\right)}{\varepsilon^2}$$

2.3 Smooth surfaces

In the following, S is assumed to be a compact smooth oriented surface without boundary embedded in $\mathbb{R}^3$. At a point $p \in S$, $N(p)$ denotes the oriented unit vector normal to S at p. $\rho_1(p)$ and $\rho_2(p)$, $\rho_1(p) \le \rho_2(p)$, denote respectively the minimum and the maximum principal (signed) curvatures at p.

Let us denote by $\rho_{\sup}$ the supremum of the absolute values of the curvatures on S :

$$\rho_{\sup} = \sup_{p\in S} \max(|\rho_1(p)|, |\rho_2(p)|)$$

S satisfies some additional *generic* conditions that are given at the beginning of Sections 5.1 and 6.1.

2.4 Empty osculating spheres and Z

Definition 2. A Delaunay sphere σ is a sphere that passes through at least two points of A and such that the interior of $B(\sigma)$ does not contain any point of A.

Definition 3. We say that a sphere $\sigma_{(c,R)}$ is *ϵ-empty* iff $\forall p \in S, d(c,p) \ge R - \epsilon$. We simply say *empty* for 0-empty.

Observe that a sphere of radius $R \le \epsilon$ is always ϵ-empty. ϵ-empty spheres will play a crucial role in our proof. As already observed in the introduction, we have in particular

LEMMA 2 (WEAK PENETRATION LEMMA.). *The Delaunay spheres of an ε-sample of S are ε-empty.*

In order to characterize ϵ-empty spheres, we introduce the *penetration map* $\omega : S \times \mathbb{R}^3 \to \mathbb{R}$:

$$\omega(p,c) = d(p,c) - \inf_{q\in S} d(q,c)$$

$\forall (p,c) \in S \times \mathbb{R}^3$, a sphere centered at c and passing through p is ϵ-empty if and only if $\omega(p,c) \le \epsilon$ and is empty if and only if $\omega(p,c) = 0$. Note that $\forall (p,c) \in S \times \mathbb{R}^3$, one has $\omega(p,c) \ge 0$.

LEMMA 3. *The penetration map ω is 3-Lipschitz and therefore continuous.*

Definition 4. We say that a sphere σ centered at c is *above* (*resp. below*) S at a point $p \in S$ iff $(c-p)\cdot N(p) \ge 0$ (*resp.* $(c-p)\cdot N(p) \le 0$).

Observation 1. Note that if the normal $N(p)$ at a point p is changed to $-N(p)$, then $\rho_1(p)$ and $\rho_2(p)$ are changed respectively to $-\rho_2(p)$ and $-\rho_1(p)$ and a sphere σ above S at p is changed to a sphere σ below S at p. Therefore, if we establish a result for a sphere σ above S at p, we obtain a similar result for a sphere σ below S at p by replacing $N(p)$, $\rho_1(p)$ and $\rho_2(p)$ by, respectively, $-N(p)$, $-\rho_2(p)$ and $-\rho_1(p)$.

Observation 2. If $\sigma_{(c,R)}$ is an empty sphere passing through $p \in S$, then one has $\frac{1}{R} \ge \rho_2(p)$ if it is above S at p, and $-\frac{1}{R} \le \rho_1(p)$ if it is below S at p.

Definition 5. We say that a sphere $\sigma_{(c,R)}$ is osculating S at $p \in S$ iff

1. $\sigma_{(c,R)}$ is tangent to the surface S at p,
2. $\frac{1}{R} = \rho_2(p)$ if the sphere is above S at p,
3. $-\frac{1}{R} = \rho_1(p)$ if the sphere is below S at p.

We now define the set $\mathbf{Z}$ which also plays a central role in our proof:

Definition 6. $\mathbf{Z}$ is the set of points $p \in S$ for which there exists an empty sphere osculating S at p.

Note that if $p \in \mathbf{Z}$, the fact that there exists an empty sphere osculating the surface at p implies that the principal curvature whose modulus is equal to $\frac{1}{R}$ is extremal at p. Hence p lies on a ridge. The converse is not true. Only those points of the ridges where the osculating spheres are empty belong to $\mathbf{Z}$ (See Figure 1).

Definition 7. Let σ be a sphere and $S' \subset S$ be a connected component of $S \cap B(\sigma)$. We say that $p_0 \in S'$ is an *anchor point* of σ in S' iff p_0 is a point of S' closest to the center of σ.

Definition 8. We say that a sphere is *bitangent* if it is tangent to S at two distinct points p and $q \neq p$. q is called a *symmetric point* of p.

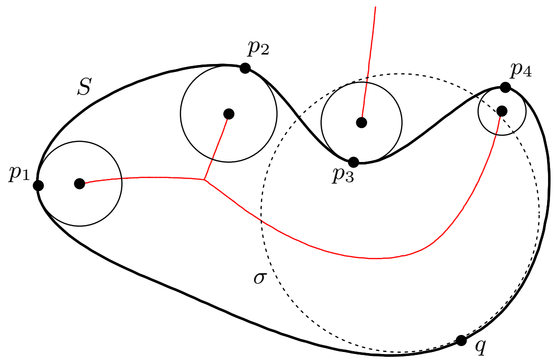

Figure 1: Points p_1, p_2, p_3 and p_4 belong to Z. The dotted circle σ osculates the curve at q but is not empty. Therefore, $q \notin$ Z.

3. OVERVIEW OF THE PROOF

The surface S is assumed to be fixed. A *constant* designates a number that depends only on S. In particular, it does not depend on the set A nor on any specific point $p \in S$.

In order to bound the complexity of the Delaunay triangulation, we count the Delaunay edges of A. Indeed, the number of Delaunay edges of A is an upper bound of the number of Delaunay tetrahedra.

Two points p and q are connected by a Delaunay edge if there exists a Delaunay sphere σ passing through p and q. An important fact concerning Delaunay spheres is that they cannot penetrate too much the surface : in fact, as stated by Lemma 2, Delaunay spheres are *ε-empty.* A stronger result will be given later (Lemma 4). Therefore, the sample points of A that are connected to p by a Delaunay edge are located in:

$$S_p = \bigcup_{\sigma\ \varepsilon\text{-empty sphere}\ \ni\ p} (S \cap B(\sigma))$$

Thanks to Lemma 1, bounding the number of Delaunay edges that are incident to p reduces to bounding the area of S_p. For generic surfaces, S_p consists of a bounded number of connected components. The area of each connected component remains small and the connected components are located near p and its symmetric points. Hence, the proof is twofold. First, we bound the area of $S \cap B(\sigma)$, where σ is an ε-empty sphere. Secondly, we study the way symmetric points of p move on the surface when p moves on the surface.

We prove that the area of $S \cap B(\sigma)$, where σ is an ε-empty sphere, is small for generic surfaces. The intersection $S \cap B(\sigma)$ may vary greatly depending on the type of surface we are considering. For instance, if we are considering a cylinder, the area of $S \cap B(\sigma)$ can be $\Omega(\varepsilon)$. This is the reason why we restrict our attention to *generic surfaces*. On generic surfaces, the area of $S \cap B(\sigma)$ is $O(\varepsilon^2)$ when we are far away from **Z**, and increases until reaching $O(\varepsilon^{\frac{3}{2}})$ as we are approaching **Z** (see Figure 2).

Since our bound on the area $S \cap B(\sigma)$ changes when we are approaching **Z**, we count the Delaunay edges in two steps. In Section 5, we prove that the number of Delaunay edges whose endpoints are far away from **Z** is $O(N)$. In Section 6, we prove that the number of Delaunay edges with an endpoint close to **Z** is $O(N \log N)$.

For technical reasons, in Sections 6 and 5, we will first count the edges for which at least one of the circumscribing Delaunay spheres has a radius bounded by some constant $R_{\max}$. By means of an inversion, we can reduce the counting of the other Delaunay edges to the counting of the edges having at least a Delaunay sphere of radii less than $R_{\max}$. Details are given in Section 7. In Section 6, we also exclude the edges for which one of the Delaunay spheres has a radius less than $R_{\min} = \frac{1}{2\rho_{\sup}}$. We count them separately in Section 8.

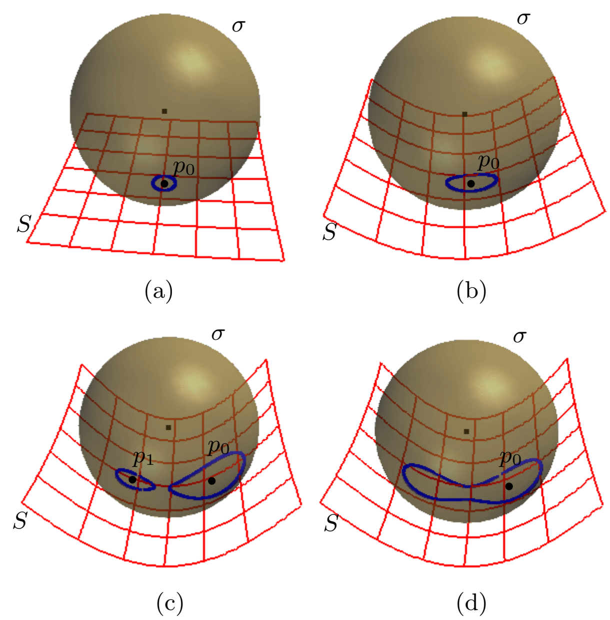

Figure 2: When a sphere σ with anchor point p_0 becomes close to the osculating sphere passing through p_0, the intersection $S \cap B(\sigma)$ stretches out in the maximal principal direction.

4. GEOMETRIC PRELIMINARIES

In this section, S is assumed to be C^3.

4.1 Local expression of the intersection between a surface and a sphere

Let σ be a sphere centered at c of radius R and p_0 an anchor point of σ. We assume σ is above S at p_0. Let $h = R - d(c, p_0)$.

For simplicity, we denote by ρ_1 and ρ_2 the principal curvatures at point p_0 instead of $\rho_1(p_0)$ and $\rho_2(p_0)$. Let us choose a reference frame as follows. We take p_0 as the origin, and the tangent plane to the surface at p_0 as the (x,y)-plane. Moreover, the x and y axes coincide with the principal directions, and the z-axis is directed along $N(p_0)$. This coordinate system is uniquely defined when $\rho_1 \neq \rho_2$ and is defined up to a rotation around the z-axis at ombilic points (where $\rho_1 = \rho_2$).

In this coordinate system, the surface S can be expressed locally as the graph $z = f_S(x, y)$ of a function f_S. The sphere σ is expressible as the graph $z = f_\sigma(x, y)$ of a function f_σ.

The intersection of S with the closed ball $B(\sigma)$ is given by

$$f = f_S - f_\sigma \geq 0.$$

The goal of this section is to give a local expression of f_S and f_σ around p_0 and to bound the difference $f = f_S - f_\sigma$.

Rather than considering the (x, y) coordinates, we prefer to use the polar coordinates (r, θ) defined by $x = r\cos\theta$ and $y = r\sin\theta$.

The Taylor expansion of f_S at p_0 is:

$$f_S(r,\theta) = \frac{1}{2}\left(\rho_2\cos^2\theta + \rho_1\sin^2\theta\right)r^2 + \eta_S(r,\theta) \tag{1}$$

where $\eta_S(r,\theta)$ represents the remainder in the Taylor expansion. Since S is C^3 and compact, there exists a constant C_S such that

$$|\eta_S(r,\theta)| \leq C_S r^3 \tag{2}$$

Consider now f_σ. For $r \leq R$, we can write:

$$f_\sigma(r,\theta) = -h + R - \sqrt{R^2 - r^2} \tag{3}$$

In order to bound $f(r,\theta)$, we consider two cases. We note $\rho = \frac{1}{R}$.

Case 1: $\rho < 2\rho_{\text{sup}}$. The Taylor expansion of f_σ in a neighbourhood of p_0 is:

$$f_\sigma(r,\theta) = -h + \frac{1}{2}\rho r^2 + \eta_\sigma(r) \tag{4}$$

where $\eta_\sigma(r)$ is the remainder in the Taylor expansion. Because $\rho < 2\rho_{\text{sup}}$, one can find a constant C (that depends only on ρ_{sup}) such that for any $r \leq \frac{R}{2}$ and any sphere σ with $\rho < 2\rho_{\text{sup}}$, one has:

$$|\eta_\sigma(r)| \leq Cr^4 \tag{5}$$

We therefore have

$$f(r,\theta) = h - \frac{1}{2}\left((\rho-\rho_2)\cos^2\theta + (\rho-\rho_1)\sin^2\theta\right)r^2 + O(r^3).$$

If we ignore the terms of orders higher than 3, the intersection $\sigma \cap S$ in the neighbourhood of p_0 is an ellipse whose half minor and major axes are respectively $\sqrt{\frac{2h}{\rho-\rho_1}}$ and $\sqrt{\frac{2h}{\rho-\rho_2}}$ (see Figure 2).

Moreover, by combining Equations 1, 2, 4 and 5, one can find a constant $\delta > 0$ such that

$$h - \frac{1}{2}(\rho-\rho_1)r^2 - \delta r^3 \leq f(r,\theta) \leq h - \frac{1}{2}(\rho-\rho_2)r^2 + \delta r^3$$

Case 2: $\rho \geq 2\rho_{\text{sup}}$. We use the fact that for $0 \leq t \leq 1$, we have $\frac{t}{2} \leq 1 - \sqrt{1-t} \leq t$, in order to bound the expression of f_σ given by Equation 3:

$$-h + \frac{1}{2}\rho r^2 \leq f_\sigma(r,\theta) \leq -h + \rho r^2$$

Therefore, one can find a constant $\delta > 0$ such that:

$$h - \frac{1}{2}(2\rho-\rho_1)r^2 - \delta r^3 \leq f(r,\theta) \leq h - \frac{1}{2}(\rho-\rho_2)r^2 + \delta r^3$$

When $\rho \geq 2\rho_{\text{sup}}$, $\frac{2\rho-\rho_1}{\rho-\rho_1} = 2 + \frac{\rho_1}{\rho-\rho_1} \leq 3$. Hence, in both cases, we have the following bounds on $f(r,\theta)$:

$$h - \frac{3}{2}(\rho-\rho_1)r^2 - \delta r^3 \leq f(r,\theta) \leq h - \frac{1}{2}(\rho-\rho_2)r^2 + \delta r^3 \tag{6}$$

4.2 The case of Delaunay spheres

An important observation in our proof is that Delaunay spheres are h-empty, with $h = \varepsilon$ (Weak penetration lemma 2). In fact, we will prove a stronger result for Delaunay spheres of radii greater than $\frac{1}{2\rho_{\text{sup}}}$. For such Delaunay spheres, $h = \Theta(\varepsilon^2)$. This result will be used in Section 5.

Let us assume that σ is a Delaunay sphere of A. From the sampling condition, there is a sample point $p = (r_p, \theta_p)$ in the geodesic disk of radius ε centered at p_0. Hence,

$$\varepsilon \geq d_S(p_0, p) \geq \|p - p_0\| \geq r_p.$$

Equation 6 then implies that for some θ_p :

$$h - \frac{3}{2}(\rho-\rho_1)r_p^2 - \delta r_p^3 \leq f(r_p,\theta_p) = 0$$

which implies in turn,

$$h \leq \frac{3}{2}(\rho-\rho_1)\varepsilon^2 + \delta\varepsilon^3. \tag{7}$$

Lemma 4 (Strong penetration Lemma.). *There exists a constant ϵ_0 such that for any ε-sample of S with $\varepsilon \leq \epsilon_0$, if σ is a Delaunay sphere of radius greater than $\frac{1}{2\rho_{\text{sup}}}$, then σ is h-empty with $h \leq 5\rho_{\text{sup}}\varepsilon^2$.*

Proof. Equation 7 yields

$$h \leq \frac{3}{2}(2\rho_{\text{sup}} + \rho_{\text{sup}})\varepsilon^2 + \delta\varepsilon^3 \leq \frac{9\rho_{\text{sup}}}{2}\varepsilon^2 + \delta\varepsilon^3.$$

With $\varepsilon \leq \frac{\rho_{\text{sup}}}{2\delta}$, we finally get $h \leq 5\rho_{\text{sup}}\varepsilon^2$. □

The next lemma uses the bound on $f(r,\theta)$ (Equation 6) in order to bound the diameter of $S \cap B(\sigma)$.

Lemma 5 (First range lemma.). *For any μ_0 and K, there exists $\epsilon_0 > 0$ such that for any ε-sample of S with $\varepsilon \leq \epsilon_0$ and any Delaunay sphere σ with anchor point p_0, we have:*

$$\mu_0 \leq \rho - \rho_2 \text{ and } \frac{\rho-\rho_1}{\rho-\rho_2} \leq K \implies \mathit{diam}\,(CC(p_0)) \leq \sqrt{7K}\,\varepsilon$$

where ρ denotes the curvature of σ, ρ_1 and ρ_2 the principal curvatures of S at p_0, and diam stands for geodesic diameter.

Proof. If we take $r \leq \frac{\mu_0}{4\delta} \leq \frac{\rho-\rho_2}{4\delta} \leq \frac{\rho-\rho_1}{4\delta}$, Equation 6 becomes

$$h - \frac{7}{4}(\rho-\rho_1)r^2 \leq f(r,\theta) \leq h - \frac{1}{4}(\rho-\rho_2)r^2.$$

Arguing as above, there exists $r_p \leq \varepsilon$ such that

$$h - \frac{7}{4}(\rho-\rho_1)r_p^2 \leq 0.$$

Hence

$$f(r,\theta) \leq \frac{7}{4}(\rho-\rho_1)\varepsilon^2 - \frac{1}{4}(\rho-\rho_2)r^2.$$

We conclude that $f(r,\theta) \leq 0$ when

$$r \geq \sqrt{7\frac{\rho-\rho_1}{\rho-\rho_2}}\,\varepsilon$$

Hence, for $r \geq \sqrt{7K}\,\varepsilon \geq \sqrt{7\frac{\rho-\rho_1}{\rho-\rho_2}}\,\varepsilon$ and $r \leq \frac{\mu_0}{4\delta}$, $f(r,\theta) \leq 0$. It is easy to see that the same conclusion holds for $r = \sqrt{7K}\,\varepsilon$ and $\varepsilon < \epsilon_0 = \frac{\mu_0}{4\sqrt{7K}\delta}$. □

4.3 Study of the ε-empty spheres

Let σ be a sphere and p an anchor point of σ. Let ρ be the curvature of σ, and $\rho_1(p)$ and $\rho_2(p)$ be the principal curvatures at point p. The first range lemma tells us that the diameter of the connected component $CC(p)$ of $B(\sigma)\cap S$ that contains p is controlled by the ratio $\sqrt{\frac{\rho-\rho_1(p)}{\rho-\rho_2(p)}}$. Therefore, in order to bound the diameter, we need to bound from below $\rho-\rho_2(p)$. Let us first observe that if σ is an empty sphere and p belongs to $\mathbf{Z}$, $\rho=\rho_2(p)$ and the ratio is undefined. Nevertheless, we prove in this section that far away from $\mathbf{Z}$, $\rho-\rho_2(p)$ can be bounded by a constant when ε is small enough (Lemma 6). This allows us to bound the size of $CC(p)$ by $O(\varepsilon)$ when p is far away from $\mathbf{Z}$ (Second range lemma 7).

Lemma 6. *For any $R_{\max}>0$ and $\delta>0$, there are $\epsilon_0>0$ and $\mu>0$ such that for any ϵ_0-empty sphere σ of curvature $\rho\geq\frac{1}{R_{\max}}$ and any point $p\in S\cap B(\sigma)$, if σ is above S at p, we have:*

$$p\in S\setminus\mathbf{Z}^{+\delta}\implies \rho-\rho_2(p)>\mu$$

Proof. Let σ be a sphere centered at c of radius R and let $p\in S\cap B(\sigma)$. We consider the sphere σ' centered at c and passing through p. Let R' be the radius of σ'. We first observe that if σ is ϵ_0-empty, then σ' is also ϵ_0-empty. Furthermore, we have the following implication:

$$\frac{1}{R'+\epsilon_0}>\rho_2(p)+\mu\implies\frac{1}{R}>\rho_2(p)+\mu$$

Therefore, it is enough to prove the following result: For any $R_{\max}>0$ and $\delta>0$, there are $\epsilon_0>0$ and $\mu>0$ such that for any ϵ_0-empty sphere $\sigma_{(c,R)}$ of radius $R\leq R_{\max}$, passing through $p\in S$ and above S at p, we have:

$$p\in S\setminus\mathbf{Z}^{+\delta}\implies\frac{1}{R+\epsilon_0}>\rho_2(p)+\mu$$

In order to establish this result, we introduce the set $\mathcal{Z}$. $\mathcal{Z}\subset S\times\mathbb{R}^3$ is the set of $(p,c)\in S\times\mathbb{R}^3$ such that the sphere centered at c and passing through p is empty and osculates the surface S at p. Intuitively, $\mathcal{Z}$ represents the set of empty osculating spheres. Observe that $\mathbf{Z}$ is the projection of $\mathcal{Z}$ on its first component. We define the set $\mathcal{F}_n\subset S\times\mathbb{R}^3$ for $n\geq1$ as follows:

$$\begin{aligned}\mathcal{F}_n=\{(p,c)\in S\times\mathbb{R}^3,\ & d(p,c)\leq R_{\max},\\ & (c-p)\cdot N(p)\geq0,\\ & \omega(p,c)\leq\frac{1}{n},\\ & \frac{1}{d(p,c)+\frac{1}{n}}\leq\rho_2(p)+\frac{1}{n}\}\end{aligned}$$

We first prove that :

$$\bigcap_{n=1}^{\infty}\mathcal{F}_n\subset\mathcal{Z}$$

Let $(p,c)\in\bigcap_{n=1}^{\infty}\mathcal{F}_n$. We note $R=d(p,c)$ and $\sigma_{(c,R)}$ the sphere centered at c and passing through p. By continuity of ω (Lemma 3), $\omega(p,c)=0$ and $\frac{1}{R}\leq\rho_2(p)$. Therefore, $\sigma_{(c,R)}$ is an empty sphere such that $\frac{1}{R}\leq\rho_2(p)$. Because an empty sphere passing through p and above S at p must satisfy $\frac{1}{R}\geq\rho_2$ (Observation 2), we have $\frac{1}{R}=\rho_2(p)$. Thus, $\sigma_{(c,R)}$ is osculating S at p and $(p,c)\in\mathcal{Z}$.

Now, by definition of $\mathcal{F}_n$, $\mathcal{F}_n$ is a compact set and $\mathcal{F}_n\supset\mathcal{F}_{n+1}$. Therefore, $\mathcal{F}_n\setminus\mathcal{Z}^{+\delta}$ is a sequence of compact sets, decreasing for the inclusion order. Because

$$\bigcap_{n=1}^{\infty}(\mathcal{F}_n\setminus\mathcal{Z}^{+\delta})=\emptyset$$

it follows (see [14], page 82) that one can find an integer k such that

$$\mathcal{F}_k\setminus\mathcal{Z}^{+\delta}=\emptyset$$

that is

$$\mathcal{F}_k\subset\mathcal{Z}^{+\delta}$$

Now assume $\sigma_{(c,R)}$ is a sphere of radius $R\leq R_{\max}$, passing through $p\in S$ and above S at p. Assume $p\in S\setminus\mathbf{Z}^{+\delta}$. Then, $(p,c)\notin\mathcal{Z}^{+\delta}$. Consequently $(p,c)\notin\mathcal{F}_k$. But since we have already imposed that $R\leq R_{\max}$, and $(c-p)\cdot N(p)\geq0$, this means that :

$$\omega(p,c)\leq\frac{1}{k}\implies\frac{1}{R+\frac{1}{k}}>\rho_2(p)+\frac{1}{k}$$

otherwise, we would have a contradiction. This allows us to conclude with $\epsilon_0=\mu=\frac{1}{k}$. $\square$

Lemma 7 (Second range lemma.). *For any $R_{\max}>0$ and $\delta>0$, there are $\epsilon_0>0$ and $K_1>0$ such that for any (ε,κ)-sample of S with $\varepsilon\leq\epsilon_0$, and any Delaunay sphere σ having an anchor point $p_0\in S\setminus\mathbf{Z}^{+\delta}$, the geodesic diameter of the connected component $CC(p_0)$ of $B(\sigma)\cap S$ containing p_0 is bounded by $K_1\varepsilon$.*

Proof. Let ρ_1 and ρ_2 denote the principal curvatures at p_0, and ρ the curvature of σ. By Lemma 2, σ is ε-empty. By Lemma 6, one can find two constants $\mu>0$ and $\epsilon_1>0$ such that if $\varepsilon\leq\epsilon_1$:

$$\mu\leq\rho-\rho_2\leq\rho-\rho_1$$

We distinguish two cases:

- If $\rho<2\rho_{\sup}$, $\mu\leq\rho-\rho_2\leq\rho-\rho_1\leq3\rho_{\sup}$ and $\frac{\rho-\rho_1}{\rho-\rho_2}\leq\frac{3\rho_{\sup}}{\mu}$.
- If $\rho\geq2\rho_{\sup}$, $\frac{\rho-\rho_1}{\rho-\rho_2}\leq3$

Hence $\frac{\rho-\rho_1}{\rho-\rho_2}\leq\max(3,\frac{3\rho_{\sup}}{\mu})$ and Lemma 5 allows to conclude. $\square$

4.4 Study of the bitangency relationship

S is parametrized by a set of n maps $\phi_i:D_i\in\mathbb{R}^2\mapsto S\subset\mathbb{R}^3$ where D_i is an open subset of $\mathbb{R}^2$ and $S=\cup_{i=1}^{n}\phi_i(D_i)$. The maps ϕ_i are C^3 and regular, that is their derivatives are of full rank (rank 2). We say that $u=(u_1,u_2)\in D_i$ is a parameter of $p\in S$ if $p=\phi_i(u)$ for some i. One can introduce $D\subset\mathbb{N}\times\mathbb{R}^2$, defined by $D=\cup_{i=1}^{n}(\{i\}\times D_i)$. D is the set of parameters of S.

For the sake of simplicity, we write $\frac{\partial S}{\partial u}$ instead of $\frac{\partial\phi_i}{\partial u}$, with

$$\frac{\partial S}{\partial u}=(\frac{\partial S}{\partial u_1},\frac{\partial S}{\partial u_2})$$

where $\frac{\partial S}{\partial u_1}$ (*resp.* $\frac{\partial S}{\partial u_2}$) is the partial derivative with respect to the first (*resp.* second) coordinate of u. We also use the notation $N(u)$, $\rho_1(u)$ and $\rho_2(u)$, for $u\in D$, in place of $N(S(u))$, $\rho_1(S(u))$ and $\rho_2(S(u))$.

Let σ be a sphere centered at c and tangent to the surface at $S(u)$ and $S(v)$. The goal of this section is to bound the displacement of $S(v)$ along S when $S(u)$ moves along S. We introduce a map Ω such that $\Omega(u, v, c) = 0$ iff the sphere centered at c is tangent to the surface at the two points $S(u)$ and $S(v)$.

Definition 9. The map $\Omega : D \times D \times \mathbb{R}^3 \mapsto \mathbb{R}^5$ is defined by $\Omega(u, v, c) = (\Omega_1(u, v, c), \Omega_2(u, v, c), \Omega_3(u, v, c))$ with
$\Omega_1(u, v, c) = (S(u) - c)^2 - (S(v) - c)^2$
$\Omega_2(u, v, c) = \frac{\partial}{\partial u}(S(u) - c)^2$
$\Omega_3(u, v, c) = \frac{\partial}{\partial v}(S(v) - c)^2$

The implicit function theorem tells us that if:

$$det_{vc}(u, v, c) = \left| \frac{\partial \Omega}{\partial v}(u, v, c) \quad \frac{\partial \Omega}{\partial c}(u, v, c) \right| \neq 0$$

then v and c can be expressed as functions of u.

LEMMA 8. *For any $R_{\max} > 0$ and $\delta > 0$, there exists a constant $\epsilon_0 > 0$ such that for any ϵ_0-empty sphere centered at c, of radius $R \leq R_{\max}$, bitangent to S at $S(u)$ and $S(v)$:*

$$S(v) \in S \setminus \mathbf{Z}^{+\delta} \implies |det_{vc}(u, v, c)| > \epsilon_0.$$

PROOF. Omitted in this extended abstract. □

We say that a point $q \in S$ is a $(R_{\max}, \epsilon, \delta)$-*standard symmetric point* of $p \in S$ if q is a symmetric point of p and belongs to $S \setminus \mathbf{Z}^{+\delta}$, and if, in addition, the sphere tangent to S at p and q is ϵ-empty and has a radius $R \leq R_{\max}$.

LEMMA 9 (LIPSCHITZ LEMMA). *For any $R_{\max} > 0$ and $\delta > 0$, there are $\epsilon_0 > 0$, $\alpha_0 > 0$ and $K_2 > 0$ such that, for any point $p_0 \in S$ with a $(R_{\max}, \epsilon_0, \delta)$-standard symmetric point* $\mathrm{Sym}(p_0)$ *and for any $p_1 \in S$ satisfying $d_S(p_0, p_1) < \alpha_0$, there exists a symmetric point* $\mathrm{Sym}(p_1)$ *of p_1 such that :*

$$d_S(\mathrm{Sym}(p_1), \mathrm{Sym}(p_0)) \leq K_2 d_S(p_1, p_0)$$

PROOF. We just sketch the proof in this extended abstract. Thanks to Lemma 8, everything is prepared to use the implicit function theorem. In a neighbourhood of u_0, one can express v and c as functions of u when constrained by the bitangency relation. In fact, thanks to the uniform lower bound on $|det_{vc}(u, v, c)|$ and compactness arguments, we get a uniform Lipschitz constant K_2 on these dependencies and a uniform bound α_0 on the size of the neighboorhoods in which this relation holds. □

5. COUNTING DELAUNAY EDGES WITH BOTH ENDPOINTS FAR AWAY FROM Z

5.1 Generic conditions

In section 6, S is assumed to be C^3 and to satisfy the following property that is true generically.

Property 1. There is a number M that uniformly bounds the number of symmetric points of any point $p \in S$.

5.2 Counting Delaunay edges

We associate to each Delaunay edge $e = (a_1 a_2)$ a smallest Delaunay sphere σ passing through its endpoints a_1 and a_2. Let $CC_\sigma(a_0)$ and $CC_\sigma(a_1)$ be the connected components of $S \cap B(\sigma)$ that contain a_0 and a_1 respectively. We call *anchor points* of e the anchor points of σ in $CC_\sigma(a_0)$ and $CC_\sigma(a_1)$. Note that a Delaunay edge may have one or two anchor points.

We denote by E_f the set of Delaunay edges whose anchor points are contained in $S \setminus \mathbf{Z}^{+\delta}$ and whose associated spheres have radii $\leq R_{\max}$. The main result of this section is to prove that $\sharp(E_f) = O(N)$ (Proposition 10). The proof relies on the second range lemma 7 and the Lipschitz lemma 9.

PROPOSITION 10. *For any $R_{\max} > 0$ and $\delta > 0$, there are two constants ϵ_0 and C such that, for any (ε, κ)-sample A with $\varepsilon \leq \epsilon_0$, $\sharp(E_f) \leq C \times N$ where N is the size of A.*

PROOF. Let $e = (a_0 a_1) \in E_f$. Let σ be the smallest Delaunay sphere associated to e, c its center. By the second range lemma 7, one can find two constants ϵ_1 and K_1 such that for $\varepsilon \leq \epsilon_1$, the geodesic diameters of $CC_\sigma(a_0)$ and $CC_\sigma(a_1)$ are at most $K_1 \varepsilon$.

If a_0 and a_1 belong to the same connected component of $S \cap B(\sigma)$, we have:

$$d_S(a_1, a_0) \leq K_1 \epsilon \tag{8}$$

Assume now that a_0 and a_1 do not belong to the same connected component of $S \cap B(\sigma)$. Let p_0 and p_1 be the anchor points of σ in $CC_\sigma(a_0)$ and $CC_\sigma(a_1)$ respectively, and assume that p_0 is closer to c than p_1. We consider the smallest sphere σ_1 centered on the segment $[cp_1]$, tangent to S at p_1 and whose intersection with $CC_\sigma(a_0)$ is not empty (see Figure 3). Let $\mathrm{Sym}(p_1)$ be a point of $\sigma_1 \cap CC_\sigma(a_0)$. Observe that σ_1, which is contained in σ, is an ε-empty sphere tangent to S at p_1 and $\mathrm{Sym}(p_1)$. By the Lipschitz lemma 9, there exists two constants ϵ_2 and K_2 such that for $\varepsilon \leq \epsilon_2$, one can find a symmetric point $\mathrm{Sym}(a_0)$ of a_0 satisfying:

$$d_S(p_1, \mathrm{Sym}(a_0)) \leq K_2 d_S(\mathrm{Sym}(p_1), a_0)$$

Thus:

$$\begin{aligned} d_S(a_1, \mathrm{Sym}(a_0)) &\leq d_S(a_1, p_1) + d_S(p_1, \mathrm{Sym}(a_0)) \\ &\leq K_1(1 + K_2)\varepsilon \end{aligned}$$

For each point a_0 in A, we consider the set $E_f(a_0)$ of all Delaunay edges $(a_0 a_1) \in E_f$ incident to a_0 with the additionnal property that the anchor point p_0 is not further from c than p_1. Note that each edge in E_f belongs to at least one set $E_f(a_0)$ for some $a_0 \in A$.

Using Property 1, we know that a_0 has at most M symmetric points. Therefore, the Delaunay edges in $E_f(a_0)$ have their other endpoints in at most M geodesic disks of radius at most $K_1(1 + K_2)\varepsilon$. By Lemma 1, one can find two constants $\epsilon_3 > 0$ and C such that the number of sample points in the union of those disks is at most C. Let $\epsilon_0 = \min(\epsilon_1, \epsilon_2, \epsilon_3)$. For $\varepsilon \leq \epsilon_0$, $\sharp(E_f(a_0)) \leq C$. Therefore the number of Delaunay edges in E_f is at most $C \times N$. □

6. COUNTING DELAUNAY EDGES WITH AN ENDPOINT CLOSE TO Z

In this section, S is assumed to be C^5.

6.1 Generic conditions

We recall that $\mathbf{Z}$ is the set of points where an empty sphere is osculating the surface. Let us consider a point $p_0 \in \mathbf{Z}$. Let ρ_2 and ρ_1 be the principal curvatures at point p_0. In this section, we no longer assume that ρ_2 is the maximal

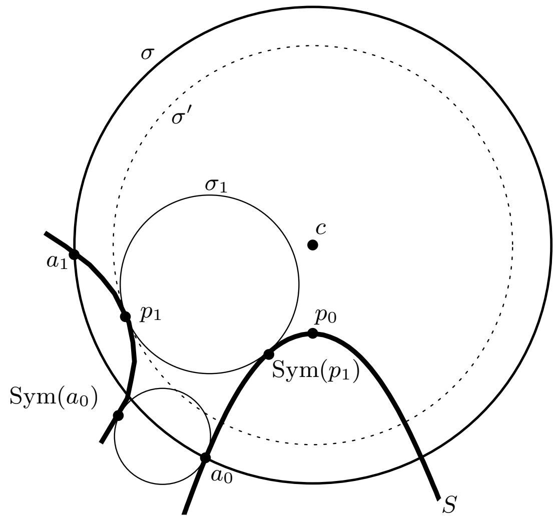

Figure 3: For the proof of Lemma 10.

principal curvature, which may be either ρ_1 or ρ_2. Following [3], we choose coordinates in $\mathbb{R}^3$ such that p_0 is at the origin, the (x, y)-plane is the tangent plane to the surface at p_0, and the principal directions coincide with the x and y axis. We choose the orientation such that $\rho_2 \geq 0$. Using almost the same notations as in [3], the surface is then expressible as the graph of the function f_S :

$$\begin{aligned} f_S(x, y) &= \frac{1}{2}(\rho_2 x^2 + \rho_1 y^2) \\ &\quad + \frac{1}{6}(ax^3 + 3bx^2y + 3cxy^2 + dy^3) \\ &\quad + \frac{1}{24}(ex^4 + 4fx^3y + 6gx^2y^2 + 4hxy^3 + iy^4) \\ &\quad + \frac{1}{120}(jx^5 + \cdots) + O(x, y)^6 \end{aligned}$$

Let σ be an empty sphere that osculates S at p_0 and is above S at p_0. In a neighbourhood of $(0, 0)$, σ can be expressed as the graph of $f_\sigma(x, y)$:

$$f_\sigma(x, y) = \frac{1}{2}\rho_2(x^2 + y^2) + \frac{1}{8}\rho_2^3(x^4 + 2x^2y^2 + y^4) + O(x, y)^5$$

The fact that σ is empty imposes that $f_S(x, y) \leq f_\sigma(x, y)$ in a neighbourhood of $(0, 0)$. Let $f = f_S - f_\sigma$. We have :

$$\begin{aligned} f(x, y) &= \frac{1}{2}(\rho_1 - \rho_2)y^2 \\ &\quad + \frac{1}{6}(ax^3 + 3bx^2y + 3cxy^2 + dy^3) \\ &\quad + \frac{1}{24}((e - 3\rho_2^3)x^4 + \cdots) + O(x, y)^5 \end{aligned} \quad (9)$$

Along the curve $x = 0$, $f(0, y) = -\frac{1}{2}(\rho_2 - \rho_1)y^2 + O(y^3)$. The condition $f(0, y) \leq 0$ imposes that $\rho_2 - \rho_1 \geq 0$. Therefore, ρ_2 must be the maximal principal curvature at p_0. Let us now assume $\rho_2 - \rho_1 = 0$. In order for equation 9 to be negative in a neighbourhood of $(0, 0)$, all third order terms should vanish, i.e. $a = b = c = d = 0$, which cannot happen generically. Since $\rho_2 - \rho_1$ is a continuous function on $\mathbf{Z}$ which is compact, we have the following generic property on $\mathbf{Z}$:

Property 2. There is a constant $\beta_0 > 0$ such that, for every point $p_0 \in \mathbf{Z}$, one has:

$$\rho_2 - \rho_1 \geq \beta_0$$

A point p_0 where $\rho_2 = \rho_1$ is called an *umbilic* point. Because of Property 2, we deduce the following generic property [3] :

Property 3. The umbilic points of S are isolated points and they do not lie on $\mathbf{Z}$.

We now look at the value of $f(x, y)$ along the curvature line $y = \phi(x)$ passing through p_0 and associated to ρ_2. Belyaev et al. [3] have established that the curvature line $y = \phi(x)$ can be approximated by the parabola $y = \frac{bx^2}{2(\rho_2 - \rho_1)}$. Let

$$\alpha = 3\rho_2^3 - e - \frac{3b^2}{\rho_2 - \rho_1} \quad (10)$$

Along the curvature line $y = \phi(x)$, $f(x, \phi(x)) = \frac{1}{6}ax^3 - \frac{1}{24}\alpha x^4 + O(x^5)$. Therefore, the condition $f(x, \phi(x)) \leq 0$ in a neighbourhood of 0 implies that $a = 0$ and $\alpha \geq 0$.

Now, if $a = 0$ and $\alpha = 0$, the fifth order coefficient in the Taylor expansion of $f(x, \phi(x))$ must vanish. We therefore have three conditions that need to be satisfied, which cannot happen generically.

We deduce the following generic property on $\mathbf{Z}$:

Property 4. There is a constant $\alpha_0 > 0$ such that, for every point $p_0 \in \mathbf{Z}$,

$$a = 0 \quad \text{and} \quad \alpha \geq \alpha_0$$

We also have:

Property 5. $\mathbf{Z}$ is made of a finite set of C^2 smooth curves whose total length l is bounded.

The boundary of each curve in $\mathbf{Z}$ is either empty or consists of two points. Consider an endpoint p of a curve of $\mathbf{Z}$ and a sphere σ osculating S at p. The sphere σ is tangent to S at another point $q(p)$ which does not lie on $\mathbf{Z}$ generically. Let $\mathbf{Y}$ be the set of all $q(p)$. Generically, $\mathbf{Y} \cap \mathbf{Z} = \emptyset$ and, since $\mathbf{Z}$ is composed of a finite number of curves, $\mathbf{Y}$ is finite.

Let σ be an empty osculating sphere passing through $z \in \mathbf{Z}$. If we slightly perturb σ, we will obtain a sphere that intersects S in at most three connected components, at most two near z and, if σ intersects $\mathbf{Y}$, at most one near $\mathbf{Y}$.

Lemma 11. *For any $\xi > 0$ and $R_{\max} > 0$, one can find two constants $\delta > 0$ and $\epsilon_0 > 0$ such that, for any ϵ_0-empty sphere σ of radius at most $R_{\max}$ and having an anchor point in $\mathbf{Z}^{+\delta}$, $S \cap B(\sigma)$ has at most three connected components. At most two of the connected components are contained in $\mathbf{Z}^{+\xi}$ and at most one is contained in $\mathbf{Y}^{+\xi}$.*

6.2 Study of the bitangency relationship

Because the curves making $\mathbf{Z}$ are compact and C^2, there is a bound on their curvature. Hence, if one chooses δ small enough, at each point $p \in \mathbf{Z}^{+\delta}$ corresponds a unique closest point $\pi(p) \in \mathbf{Z}$.

Let us choose a reference frame. We take $\pi(p)$ as the origin, and the tangent plane to the surface at $\pi(p)$ as the (x,y)-plane, the y axis being tangent to $\mathbf{Z}$. The orthogonal projection in the xy-plane gives a local parametrization of S. Note that in this local parametrization, because the segment $[p\pi(p)]$, is orthogonal to $\mathbf{Z}$ at $\pi(p)$, one has $p = S(x_p, 0)$.

We denote by $p' = S(x'_p, y'_p)$ a symmetric point of p. We have:

$$(N(x_p, 0) - N(x'_p, y'_p)) \times (S(x_p, 0) - S(x'_p, y'_p)) = 0 \quad (11)$$

We search a Taylor expansion of x'_p and y'_p with respect to x_p. In the full version of the paper, we will show that:

$$x'_p = -x_p + b' x_p^2 + O(x_p^3)$$
$$y'_p = c' x_p^3 + O(x_p^4)$$

Writing a Taylor expansion of Equation 11 and equating this Taylor expansion to 0 allows us to determine the coefficients b' and c' in the Taylor expansions of x'_p and y'_p. We obtain that:

LEMMA 12. *One can find a constant δ such that, for any point $p \in \mathbf{Z}^{+\delta}$, p has a symmetric point $p' \in \mathbf{Z}^{+2\delta}$. If $(x_p, 0)$ and (x'_p, y'_p) are the coordinates of p and p' in the local frame at $\pi(p)$, we have:*

$$x'_p = -x_p + \left(\frac{j}{5\alpha} + \frac{bf}{(\rho_2 - \rho_1)\alpha}\right) x_p^2 + O(x_p^3)$$
$$y'_p = O(x_p^3)$$

p' is called the local symmetric point *of p.*

We denote by $\sigma_0(p)$ the maximal empty sphere above S at p. Let $\rho_0(p)$ be the curvature of $\sigma_0(p)$. We have:

$$\rho_0(p) = \frac{\|N(p) - N(p')\|}{\|p - p'\|} \quad (12)$$

By reporting the Taylor expansion of x'_p and y'_p in the above equation, we obtain a Taylor expansion of $\rho_0(p)$:

LEMMA 13. *A Taylor expansion of $\rho_0(p)$ at 0 is:*

$$\rho_0(p) = \rho_2 - \left(\frac{\alpha}{6} + \frac{b^2}{2(\rho_2 - \rho_1)}\right) x_p^2 + O(x_p^3)$$

We also recall the Taylor expansions of $\rho_2(p)$ and $\rho_1(p)$, which can be found in [3]:

LEMMA 14. *The Taylor expansions of $\rho_2(p)$ and $\rho_1(p)$ at 0 are:*

$$\rho_2(p) = \rho_2 - \left(\frac{\alpha}{2} + \frac{b^2}{2(\rho_2 - \rho_1)}\right) x_p^2 + O(x_p^3)$$
$$\rho_1(p) = \rho_1 + O(x_p)$$

Let us observe that $\rho_0(p) - \rho_2(p) = \frac{1}{3}\alpha x_p^2 + O(x_p^3)$.

6.3 Local expression of the intersection between a surface and a sphere

We call *closest anchor point* of a sphere σ, the anchor point which is closest to the center of σ.

Let p be the closest anchor point of an ε-empty sphere σ. In order to study the intersection between the surface S and σ, we need to compare the curvature ρ of σ to the principal curvatures $\rho_2(p)$ and $\rho_1(p)$ of S at p. In Section 4.3, thanks to the fact that p was far from $\mathbf{Z}$, we were able to bound from below the two differences $\rho - \rho_2(p)$ and $\rho - \rho_1(p)$ by $\Omega(1)$ (Lemma 6). As a consequence, the connected component of $S \cap B(\sigma)$ containing p had size $O(\varepsilon)$ (Second range lemma 7).

In this section, since p is close to $\mathbf{Z}$, ρ is close to $\rho_2(p)$ and the difference $\rho - \rho_2(p)$ cannot be bounded from below by a constant anymore. Nevertheless, thanks to our generic conditions, we can prove that $\rho - \rho_2(p)$ is bounded from below by $\Omega(x_p^2)$ and $\rho - \rho_1(p)$ by $\Omega(1)$. Therefore, we are able to bound the size of the connected component of $S \cap B(\sigma)$ containing p (Third range lemma 16).

We start this section by relating the curvature ρ of σ to the curvature $\rho_0(p)$ of a maximal empty sphere $\sigma_0(p)$ containing σ and for which a Taylor expansion of the curvature is known. Let c_0 and R_0 denote the center and the radius of the maximal empty sphere $\sigma_0(p)$, and $\sigma_\varepsilon(p)$ the sphere centered at c_0 of radius $R_0 + 5\rho_{\sup}\varepsilon^2$.

LEMMA 15. *There exists a constant ϵ_0 such that, for any ε-sample of S with $\varepsilon \le \epsilon_0$, any Delaunay sphere σ of radius greater than $\frac{1}{2\rho_{\sup}}$ is included in $B(\sigma_\varepsilon(p))$ where p is the closest anchor point of σ (see Figure 4).*

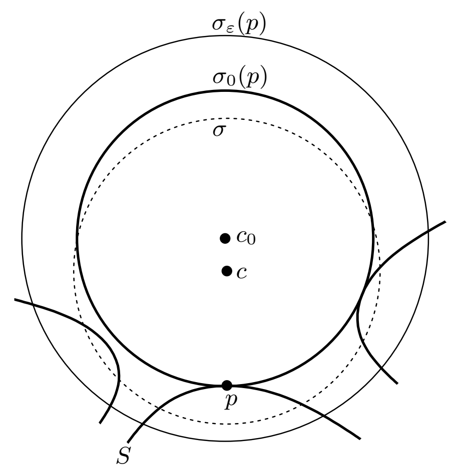

Figure 4: A sphere σ with closest anchor point p is contained in $B(\sigma_\varepsilon(p))$.

PROOF. Follows from the strong penetration lemma 4. ☐

We now study $S \cap B(\sigma_\varepsilon(p))$. Observe that p is an anchor point of $\sigma_\varepsilon(p)$. From Lemma 11, we know that if p is sufficiently close to $\mathbf{Z}$ and ε is small enough, $S \cap B(\sigma_\varepsilon(p))$ is composed of at most three connected components, among which at most two are close to $\mathbf{Z}$. In order to describe the shape of the connected components close to $\mathbf{Z}$, we define the set $\Sigma(p, w, h)$ as follows. Let $T_1(p)$ and $T_2(p)$ be the principal directions associated to the principal curvatures $\rho_1(p)$ and $\rho_2(p)$. We consider the set of points q of S for which $|(q - p) \cdot T_2(p)| \le w$ and $|(q - p) \cdot T_1(p)| \le h$ and define $\Sigma(p, w, h)$ as the connected component containing p. Roughly speaking, $\Sigma(p, w, h)$ resembles a rectangle with sides w and h aligned with the principal directions of S at point p.

LEMMA 16 (THIRD RANGE LEMMA). *For δ small enough, one can find two constants C_0, C_1 such that for ε small enough, for any point $p \in \mathbf{Z}^{+\delta}$, the intersection $\mathbf{Z}^{+2\delta} \cap$*

$B(\sigma_\varepsilon(p))$ is contained in the set $\Sigma^\varepsilon(p)$ defined as follows:

$$\Sigma\left(p, \frac{C_0^2\varepsilon}{|x_p|}, C_1\varepsilon\right) \cup \Sigma\left(p', \frac{C_0^2\varepsilon}{|x_p|}, C_1\varepsilon\right), \text{ for } C_0\sqrt{\varepsilon} \le |x_p| \le \delta$$

$$\Sigma\left(\pi(p), 2C_0\sqrt{\varepsilon}, C_1\varepsilon\right), \text{ for } |x_p| \le C_0\sqrt{\varepsilon}$$

where p' is the local symmetric point of p and $p = S(x_p, 0)$ in the coordinate system introduced in Section 6.2.

This result is obtained by bounding the Taylor expansion of the difference between S and $\sigma_\varepsilon(p)$. For this, we need the expression of $\rho_0(p)$, $\rho_1(p)$ and $\rho_2(p)$ given in Lemma 13 and Lemma 14. For $C_0\sqrt{\varepsilon} \le |x_p| \le \delta$, we use the coordinate system centered at p and aligned with the principal directions at p. For $|x_p| \le C_0\sqrt{\varepsilon}$, we use the coordinate system centered at $\pi(p)$ and aligned with the principal directions at $\pi(p)$.

As in Section 5.2, we associate to each Delaunay edge a smallest Delaunay sphere. Let $E(p)$ be the set of Delaunay edges incident to $p \in A$ and for which the smallest Delaunay sphere has a radius less than $R_{\max}$ and greater than $R_{\min}$.

LEMMA 17. *For δ small enough, there are constants C_0, K_1 and K_2 such that, for ε small enough, if $p \in A \cap \mathbf{Z}^{+\delta}$, the number of Delaunay edges incident to p and belonging to $E(p)$ is bounded by:*

$$\begin{array}{ll} K_1\varepsilon^{-\frac{1}{2}} & \text{if} \quad |x_p| \le C_0\sqrt{\varepsilon} \\ \dfrac{K_2}{|x_p|} & \text{if} \quad C_0\sqrt{\varepsilon} \le |x_p| \le \delta \end{array}$$

where $p = S(x_p, 0)$ in the coordinate system introduced in Section 6.2.

PROOF. The proof is omitted in this extended abstract. It uses the Lipschitz lemma 9, Lemma 11, Lemma 15 and the third range lemma 16. ☐

6.4 Counting Delaunay edges

As in Section 5.2, we associate to each Delaunay edge e the smallest Delaunay sphere σ passing through its endpoints. The *anchor points* of $e = (a_1a_2)$ are the anchor points of σ in the connected components of $S \cap B(\sigma)$ that contains a_0 and a_1.

Let $R_{\min} = \frac{1}{2\rho_{\sup}}$. We denote by E_s the set of Delaunay edges that have at least one anchor point in $\mathbf{Z}^{+\delta}$ and whose associated spheres have a radius $R_{\min} \le R \le R_{\max}$. Note that $E_f \cup E_s$ contains all the Delaunay edges associated to spheres of radius $R_{\min} \le R \le R_{\max}$.

LEMMA 18. *For δ small enough, there is a constant C such that, for ε small enough, the number of Delaunay edges having at least one endpoint in $\mathbf{Z}^{+\delta}$ is bounded by:*

$$CN \log N$$

PROOF. Let C_0, K_1 and K_2 be the constants of Lemma 17. Let us define the sets Z_i, for $i \in \{1, \ldots, n = \lceil \frac{\delta}{C_0\varepsilon^{\frac{1}{2}}} \rceil\}$:

$$Z_1 = \mathbf{Z}^{+C_0\varepsilon^{\frac{1}{2}}}$$

$$Z_i = \mathbf{Z}^{+iC_0\varepsilon^{\frac{1}{2}}} \setminus \mathbf{Z}^{+(i-1)C_0\varepsilon^{\frac{1}{2}}}$$

One has :

$$\mathbf{Z}^{+\delta} \subset \bigcup_{i=1}^{n} Z_i$$

Let us count first the number of Delaunay edges having an endpoint in $Z_1 = \mathbf{Z}^{+C_0\varepsilon^{\frac{1}{2}}}$.

Note that, for some constant C:

$$\sharp(A \cap \mathbf{Z}^{+C_0\varepsilon^{\frac{1}{2}}}) \le C\frac{\varepsilon^{\frac{1}{2}}}{\varepsilon^2} = C\varepsilon^{-\frac{3}{2}}$$

and, more generally:

$$\sharp(A \cap Z_i) \le C\varepsilon^{-\frac{3}{2}}$$

We deduce from Lemma 17 an upper bound on the number of Delaunay edges having a point in Z_1 :

$$C\varepsilon^{-\frac{3}{2}} \times K_1\varepsilon^{-\frac{1}{2}} = CK_1\varepsilon^{-2} = O(N) \tag{13}$$

We count now the number of Delaunay edges having an endpoint in Z_i, for $i \in \{2, \ldots, n\}$.

Note that for $a < 1$ and δ small enough, if $S(x, y) \in Z_i$, one has $x \ge a(i-1)C_0\varepsilon^{\frac{1}{2}}$.

Let us take $a = \frac{1}{2}$:

$$S(x, y) \in Z_i \implies x \ge \frac{1}{2}(i-1)C_0\varepsilon^{\frac{1}{2}}$$

Lemma 17 then gives an upper bound on the number of Delaunay edges having an endpoint in Z_i, for $i \in \{2, \ldots, n\}$:

$$\begin{aligned} \sum_{i=2}^{n} C\varepsilon^{-\frac{3}{2}} \times \frac{2K_2}{(i-1)C_0\varepsilon^{\frac{1}{2}}} & \\ &= \frac{2CK_2}{C_0}\varepsilon^{-2}\sum_{i=2}^{n}\frac{1}{i-1} \\ &\le \frac{2CK_2}{C_0}\varepsilon^{-2}(1 + \log(n-1)) \\ &= O(N \log N) \end{aligned} \tag{14}$$

By adding the number of edges in equations 13 and 14 we get the result. ☐

PROPOSITION 19. *For any $R_{\max} > 0$ and $\delta > 0$, there are two constants ϵ_0 and C such that, for any (ε, κ)-sample A of S with $\varepsilon \le \epsilon_0$, $\sharp(E_s) \le C \times N \log N$ where N is the size of A.*

PROOF. By Lemmas 16 and 12, if an anchor point of e belongs to $\mathbf{Z}^{+\delta}$, then the endpoints of e belongs to $\mathbf{Z}^{+2\delta}$. The previous lemma then allows to conclude. ☐

7. THE CASE OF TETRAHEDRA WITH CIRCUMRADII GREATER THAN $R_{\max}$

Let $\mathcal{D}$ be the set of Delaunay balls in the Delaunay triangulation of A whose radii are strictly greater than $R_{\max}$. For a sufficiently large $R_{\max}$, all balls in $\mathcal{D}$ are above S. Moreover, because of Lemma 2, the balls in $\mathcal{D}$ cannot penetrate by more than ε into the region $\mathcal{R}$ bounded by S. It follows that there exists a point O and constants ε_0 and ρ such that, for any $\varepsilon \le \varepsilon_0$, the ball $B(O, \rho)$ does not intersect the balls of $\mathcal{D}$. Moreover, since S is compact, there exists another constant δ such that the ball $B(O, \delta)$ contains S.

We take the point O defined above as the origin and consider the inversion I of center O and ratio 1, i.e. the image of $x \neq O$ is the point $x' = \frac{x}{\|x\|^2}$. Let S' and A' be the images by I of S and A.

It is easily checked that the image B' by I of a ball B of $\mathcal{D}$ is a Delaunay ball B' of A'. Moreover, the radius of B' is at most $\frac{1}{\rho}$.

To be able to extend the results of Sections 6 and 5 to any Delaunay sphere, it remains to show that A' is a good sample of S'.

Lemma 20. *For any $\varepsilon \le \varepsilon_0$, A' is a (ε', κ')-sample of S', with $\varepsilon' = \frac{\varepsilon}{\rho^2}$ and $\kappa' = \frac{8\kappa}{\gamma^3 \rho^6}$, where $\gamma = \frac{1}{\delta^2} - \frac{\varepsilon_0^2}{\rho^4}$.*

Proof. For any point $x \in S$, there exists a point $p \in A$ such that $d(x,p) \le \varepsilon$. Hence, for any $x' \in S'$, we have

$$\|x' - p'\|^2 = \|\frac{x}{\|x\|^2} - \frac{p}{\|p\|^2}\|^2 = \frac{\|x-p\|^2}{\|x\|^2\|p\|^2} \le \frac{\varepsilon^2}{\rho^4}$$

which shows that A' is a $\frac{\varepsilon}{\rho^2}$-sample of S'.

Consider the ball B' centered at $x' \in S'$ of radius $\varepsilon' = \frac{\varepsilon}{\rho^2}$ and its image B by the inversion I. The radius r of B satisfies

$$r \le \frac{\varepsilon'}{\|x'\|^2 - \varepsilon'^2} \le \frac{\varepsilon'}{\gamma}$$

where $\gamma = \frac{1}{\delta^2} - \frac{\varepsilon_0^2}{\rho^4}$. Let us pack as many disjoint balls of radius $\frac{\varepsilon}{2}$ as possible inside B. Let b be the number of balls in the packing. We have

$$b \le \frac{8r^3}{\varepsilon^3} \le \frac{8\varepsilon'^3}{\gamma^3\varepsilon^3} \le \frac{8}{\gamma^3\rho^6}$$

A covering of B is obtained by replacing the balls in the packing by balls of radius ε instead of $\frac{\varepsilon}{2}$. Therefore, since A is an (ε, κ)-sample, B contains at most $\frac{8\kappa}{\gamma^3\rho^6}$ points of A. By the properties of the inversion, the points of A' that belong to B' are the images by I of the points of A that belong to B. This concludes the proof. $\square$

8. THE CASE OF TETRAHEDRA WITH A CIRCUMRADIUS LESS THAN $R_{\min}$

We denote by E_m the set of edges in the Delaunay triangulation of A for which one of the Delaunay spheres has a radius less than $R_{\min} = \frac{1}{2\rho_{\sup}}$.

Proposition 21. *There are two constants ϵ_0 and C such that, for any (ε, κ)-sample A of S with $\varepsilon \le \epsilon_0$, we have $\sharp(E_m) \le C \times N$, where N is the size of A.*

Proof. Omitted in this extended abstract. $\square$

9. CONCLUSION

We sum up the results stated in Propositions 10, 18, 20 and 21 in the following theorem.

Theorem 22. *Let S be a C^5 surface satisfying the genericity properties of Sections 5.1 and 6.1, and let A be an (ε, κ)-sample of S of size N. The combinatorial complexity of the Delaunay triangulation of A is $O(N \log N)$.*

An interesting special case occurs when S is the boundary of a finite set of disjoint smooth convex objects. If we only consider the subset T of the Delaunay triangulation of A consisting of the tetrahedra that are outside the convex objects, we only need to consider the case of Section 5. It follows that the combinatorial complexity of T is $O(N)$.

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