Complexity of the Delaunay Triangulation of Points on Polyhedral Surfaces

Dominique Attali * Jean-Daniel Boissonnat †

September 24, 2002

Abstract

It is well known that the complexity of the Delaunay triangulation of n points in $\mathbb{R}^d$, i.e. the number of its simplices, can be $\Omega(n^{\lceil \frac{d}{2} \rceil})$. In particular, in $\mathbb{R}^3$, the number of tetrahedra can be quadratic. Differently, if the points are uniformly distributed in a cube or a ball, the expected complexity of the Delaunay triangulation is only linear. The case of points distributed on a surface is of great practical importance in reverse engineering since most surface reconstruction algorithms first construct the Delaunay triangulation of a set of points measured on a surface.

In this paper, we bound the complexity of the Delaunay triangulation of points distributed on the boundary of a given polyhedron. Under a mild uniform sampling condition, we provide deterministic asymptotic bounds on the complexity of the 3D Delaunay triangulation of the points when the sampling density increases. More precisely, we show that the complexity is $O(n^{1.8})$ for general polyhedral surfaces and $O(n\sqrt{n})$ for convex polyhedral surfaces.

Our proof uses a geometric result of independent interest that states that the medial axis of a surface is well approximated by a subset of the Voronoi vertices of the sample points.

1 Introduction

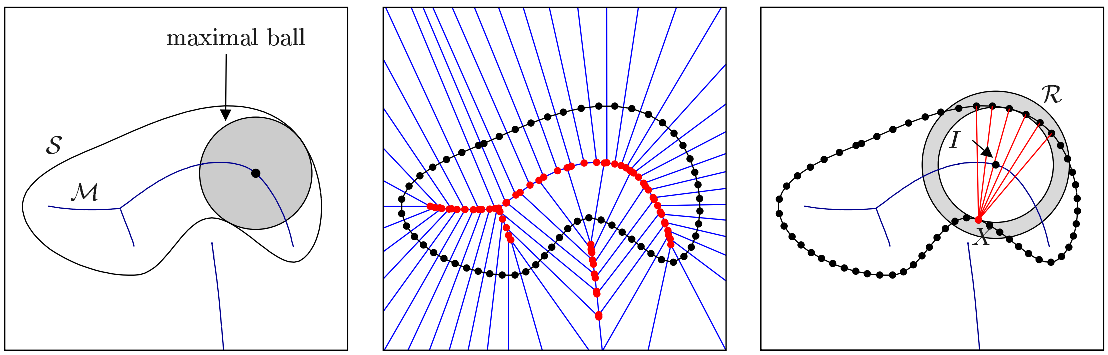
It is well known that the complexity of the Delaunay triangulation of n points in $\mathbb{R}^d$, i.e. the number of its simplices, can be $\Omega(n^{\lceil \frac{d}{2} \rceil})$. In particular, in $\mathbb{R}^3$, the number of tetrahedra can be quadratic. Differently, if the points are uniformly distributed in a cube or a ball, the expected complexity of the Delaunay triangulation is only linear [8, 9].

The case of points distributed on a surface is of great practical importance in reverse engineering since most surface reconstruction algorithms first construct the Delaunay triangulation of a set of points measured on a surface, see e.g. [2, 5]. The time complexity of those methods therefore crucially depends on the complexity of the triangulation of points scattered over a surface in $\mathbb{R}^3$. Moreover, since output-sensitive algorithms are known for computing Delaunay triangulations [6], better bounds on the complexity of the Delaunay triangulation would immediately imply improved bounds on the time complexity of computing the Delaunay triangulation.

A first result has been recently obtained by Golin and Na [11]. They proved that the expected complexity of 3D Delaunay triangulations of random points on *convex* polytopes is $\Theta(n)$. The case of points on a cylinder has been considered by J. Erickson who proved that, even if the cylinder is well-sampled, the complexity of the Delaunay triangulation may be $\Omega(n\sqrt{n})$ [10]. Erickson's paper contains also lower bounds for contrived surfaces with a non bounded ratio between diameter and minimum local feature size, a case we exclude here.

*LIS, ENSIEG, Domaine Universitaire, BP 46, 38402 Saint Martin d'Hères Cedex. e-mail : Dominique.Attali@inpg.fr
†INRIA 2004 Route des Lucioles, BP 93, 06904 Sophia-Antipolis, France. e-mail : Jean-Daniel.Boissonnat@sophia.inria.fr

In this paper, we consider the case of points distributed on the boundary of a given polyhedron. Under a mild uniform sampling condition, we provide deterministic asymptotic bounds on the complexity of the 3D Delaunay triangulation of the points when the sampling density increases. More precisely, we show that the complexity is $O(n^{1.8})$ for general polyhedral surfaces and $O(n\sqrt{n})$ for convex polyhedral surfaces. The intuition behind our result is the following. When a surface $\mathcal{S}$ is well-sampled, the circumcenters of the Delaunay simplices with a long edge are close to the medial axis $\mathcal{M}$ of $\mathcal{S}$ (see Figure 1b). It follows that the Delaunay neighbours of a point X that are sufficiently far away from X lie in a small region $\mathcal{R}$ between two spheres centered at a point I of $\mathcal{M}$ (see Figure 1c). In the case of a polyhedral surface, the intersection of $\mathcal{R}$ and $\mathcal{S}$ is contained in a bounded number of small disks and therefore X can only have a small number of Delaunay neighbours.



(a) (b) (c)

Figure 1: Overview of the proof in $\mathbb{R}^2$. (a) A curve $\mathcal{S}$ and its medial axis $\mathcal{M}$. (b) Voronoi graph of sample points. (c) The inner Delaunay neighbours of X that are far away from X lie in a small region $\mathcal{R}$.

2 Medial axis approximation

Our combinatorial bound on the complexity of the Delaunay triangulation of a light uniform sample $\mathcal{A}$ is based on a geometric result (Theorem 10 below) that states that the medial axis of the surface is well approximated by a subset of the Voronoi vertices of $\mathcal{A}$. Before stating and proving this result, we recall the definition of the medial axis of a surface and define uniform samples. In this section, we do not impose the surface to be polyhedral.

2.1 Definitions

The medial axis has first been introduced by Blum in the field of image analysis as a tool for shape description. Since then, the medial axis has been intensively studied. In this section, we recall definitions and properties related to the medial axis. Other results on the medial axis can be found in [13, 7, 12, 14, 3].

The medial axis of a subset $\mathcal{F}$ of $\mathbb{R}^3$ can be defined using the concept of a maximal ball.

Definition 1 (Maximal ball) *Let $\mathcal{F} \subset \mathbb{R}^3$. An open ball B is said to be* maximal *in $\mathcal{F}$ if and only if for any open ball B', $B \subseteq B' \subseteq \mathcal{F} \Longrightarrow B = B'$.*

Definition 2 (Medial axis of an object) *Let $\mathcal{F} \subset \mathbb{R}^3$. The* medial axis *of $\mathcal{F}$ is the closure of the centers of the maximal balls of $\mathcal{F}$.*

This definition can be extended to surfaces as follows.

Definition 3 (Medial axis of a surface) *Let $\mathcal{S}$ be an embedded two-manifold. We call medial axis of $\mathcal{S}$ the medial axis of $\mathbb{R}^3 \setminus \mathcal{S}$.*

At any point $X \in \mathcal{S}$, we associate the local feature size. The concept of local feature size has first been introduced in the context of surface reconstruction by Amenta and Bern [2].

Definition 4 (Local feature size) *The local feature size* $\mathrm{lfs}(X)$ *at a point* $X \in \mathcal{S}$ *is the distance from* X *to the medial axis of* $\mathcal{S}$.

In the most general case, there can be an infinite number of maximal balls through a given point $X \in \mathcal{S}$. Let R be the radius of any maximal ball through X:

$$\mathrm{lfs}(X) \leq R$$

However, if the normal to $\mathcal{S}$ at X is defined, there are exactly two maximal balls through X, one on each side of the tangent plane to $\mathcal{S}$ at X.

We distinguish two types of points on $\mathcal{S}$, singular and regular points:

Definition 5 (Singular and regular points) *A point* $X \in \mathcal{S}$ *is said to be* regular *iff 1. the normal to* $\mathcal{S}$ *at* X *is defined, 2. the two maximal balls through* X *touches* $\mathcal{S}$ *in at least two distinct points* X *and* $Y \neq X$*. A point is said to be* singular *iff it is not regular.*

For polyhedral surfaces, the two conditions are equivalent. The singular points are the points of the edges of the polyhedral surface. For smooth surfaces, the normal is defined everywhere and the singular points are the points of the ridges of the surface [3].

2.2 Uniform samples

Our result holds on a sampling condition that roughly says that the sample should be uniform but not arbitrarily dense locally. This is make precise through the following definitions.

Definition 6 (Uniform ε-sample) *A set of points* $\mathcal{A} \in \mathcal{S}$ *is called a* uniform ε-sample *of* $\mathcal{S}$ *iff for every point* $X \in \mathcal{S}$*, the ball* $B(X, \varepsilon)$ *contains at least one point of* $\mathcal{A}$.

Definition 7 (Light uniform ε-sample) *A uniform ε-sample is said to be* light *iff for every point* $X \in \mathcal{S}$*, the ball* $B(X, r)$ *contains* $O(\frac{r^2}{\varepsilon^2})$ *points of* $\mathcal{A}$.

Observe that our definition of a light uniform ε-sample does not impose any lower bound on the minimal distance between two sample points.

Amenta and Bern have introduced a different definition of an ε-sample [2]. The originality of their definition is to enforce the sample to fit locally the surface shape. According to their definition, point density is high where the surface has high curvature or where the object or its complement is thin. However, if the local feature size vanishes, an ε-sample, as defined in [2], will have an infinite number of points, which is not satisfactory for our purpose.

Erickson has introduced a notion of uniform sample that is related to our notion of light uniform sample but forbids points to be too close (which our definition allows) [10].

In the rest of the paper, $\mathcal{A}$ denotes a light uniform ε-sample and $\mathcal{S}$ a compact fixed surface. In particular, we assume that quantities like the area or the diameter of $\mathcal{S}$ are fixed and do not depend on ε. Hence the number of points n of $\mathcal{A}$ is bounded and $n = O\left(\frac{1}{\varepsilon^2}\right)$. We provide asymptotic results when the sampling density increases, i.e. when ε tends to 0.

2.3 Medial axis approximation

The goal of this section is to prove that the circumcenters of the Delaunay tetrahedra with long edges converge towards the medial axis (Theorem 10). Our result improves on related results obtained by

Amenta and Kolluri [1] and Boissonnat and Cazals [4]. In [1], the convergence is established for a subset of the Voronoi vertices called the *poles*. The poles of a sample point X are the two vertices of its Voronoi cell farthest from X one on either side of the surface. In this paper, we establish a result for a larger class of Voronoi vertices. We also allow the normal to the surface not to be defined everywhere in order to later be able to apply this result to polyhedral surfaces.

In this section as well as in the rest of the paper, the notation $f \ll g$ (or $f = o(g)$) means that $\frac{f}{g}$ tends to 0 when ε tends to 0. The notation $f \approx g$ means that $f - g \ll g$. The notation $f \lesssim g$ means that $f \leq g$ or $f \approx g$.

We start with a technical lemma. In this lemma, we consider four points X, A, I and V and establish a bound on $||VI||$ assuming that $||XA||$ is large enough. X and A will later designate two sample points on the surface that are adjacent in the Delaunay triangulation. I will designate the center of one of the two maximal balls through X. V will designate a vertex of the Voronoi facet dual to the Delaunay edge $[XA]$.

Lemma 8 *Let ρ and θ be two positive functions of ε such that $\rho \ll 1$ and $\theta \ll 1$. Let A, X, I and V be four points satisfying the following conditions:*

1. $||VA|| = ||VX||$,
2. $||XI|| = R$,
3. $||AI|| = R(1+\rho)$,
4. $\angle(\overrightarrow{XI}, \overrightarrow{XV}) = \theta$,
5. $||XA|| = 2l$.

If $\frac{\rho R^2}{l^2} \ll 1$ and $\frac{R\theta}{l} \ll 1$, then we have

$$||VI|| \lesssim \frac{\theta R^2}{l} + \frac{R^3 \rho}{2l^2}.$$

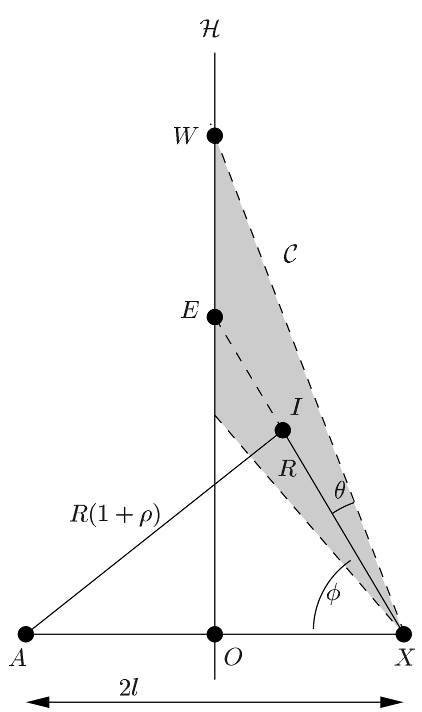


Figure 2: For the proof of Lemma 8.

Proof. Refer to Figure 2. Assume the three points A, X and I are given. Let $\mathcal{H}$ be the plane that bisects the points X and A. Let $\mathcal{C}$ be the circular cone generated by the lines passing through X and forming an angle θ with the line (XI). Point V lies in plane $\mathcal{H}$ and on cone $\mathcal{C}$. Therefore, V lies on the ellipse $\mathcal{H} \cap \mathcal{C}$. Let W be the point on this ellipse which is the farthest from point I. In addition, let O

be the mid-point of segment $[AX]$ and E be the intersection point of line (XI) with $\mathcal{H}$. Let us bound $||WI||$. By triangle inequality:

$$||WI|| \leq ||WE|| + ||EI||$$

Let ϕ be the angle between the two vectors $\overrightarrow{XO}$ and $\overrightarrow{XI}$. Since the lengths of the three sides of triangle (AXI) are known, we have:

$$\begin{aligned}\cos\phi &= \frac{-\rho(2+\rho)R^2 + 4l^2}{4lR} \\ &\approx \frac{l}{R}\end{aligned}$$

Using the expression of $\cos\phi$, we can bound $||WE||$.

$$\begin{aligned}||WE|| &= l\left(\tan(\phi+\theta) - \tan\phi\right) \\ &= \frac{l\sin\theta}{\cos(\phi+\theta)\cos\phi} \\ &\approx \frac{\theta R^2}{l}\end{aligned}$$

Let us now bound $||EI||$.

$$\begin{aligned}||EI|| &= \frac{l}{\cos\phi} - R \\ &= \frac{4l^2R}{4l^2 - \rho(2+\rho)R^2} - R \\ &\approx \frac{R^3\rho}{2l^2}\end{aligned}$$

Finally, we get:

$$||WI|| \lesssim \frac{\theta R^2}{l} + \frac{R^3\rho}{2l^2}$$

□

Let V denote a vertex of the Voronoi cell of X in the Voronoi diagram of $\mathcal{A}$. Let I be the center of one of the two maximal balls through X. By slightly adapting a result of Amenta and Bern [2, Lemma 5], we can bound $\theta = \angle(\overrightarrow{XI}, \overrightarrow{XV})$.

Lemma 9 *Let X be a regular point of $\mathcal{S}$. Assume $||VX|| \geq \varepsilon$ and $\mathrm{lfs}(X) \geq \varepsilon$. Then, the angle at X between the normal to $\mathcal{S}$ at X and the vector to V (oriented so that the angle is acute) is at most* $\arcsin\left(\frac{\varepsilon}{||VX||}\right) + \arcsin\left(\frac{\varepsilon}{\mathrm{lfs}(X)}\right)$.

The next proposition states that the circumcenters of the Delaunay tetrahedra with long edges converge towards the medial axis when ε tends to 0.

Theorem 10 (Medial axis approximation) *Let $\mathcal{A}$ be a uniform ε-sample of $\mathcal{S}$ and $X \in \mathcal{A}$ be a regular point of $\mathcal{S}$. Let $[XA]$ be a Delaunay edge and V a vertex in the Voronoi facet dual to the Delaunay edge $[XA]$. Let $B(I,R)$ be the one maximal ball through X such that $\overrightarrow{XV} \cdot \overrightarrow{XI} > 0$. Let Y be a point of $\partial B(I,R) \cap \mathcal{S}$ distinct from X. Assume that:*

1. $\varepsilon \ll \mathrm{lfs}(X)$,
2. $\frac{||XY||^2}{4R^2} \gg \frac{\varepsilon}{\mathrm{lfs}(X)}$,
3. $\frac{||XA||^2}{4R^2} \gg \frac{\varepsilon}{\mathrm{lfs}(X)}$.

Then:

$$||VI|| \lesssim R\frac{\varepsilon}{\mathrm{lfs}(X)} \max\left(\frac{10R^2}{||XY||^2}, \left(1 + \frac{2R}{||XA||}\right)^2\right).$$

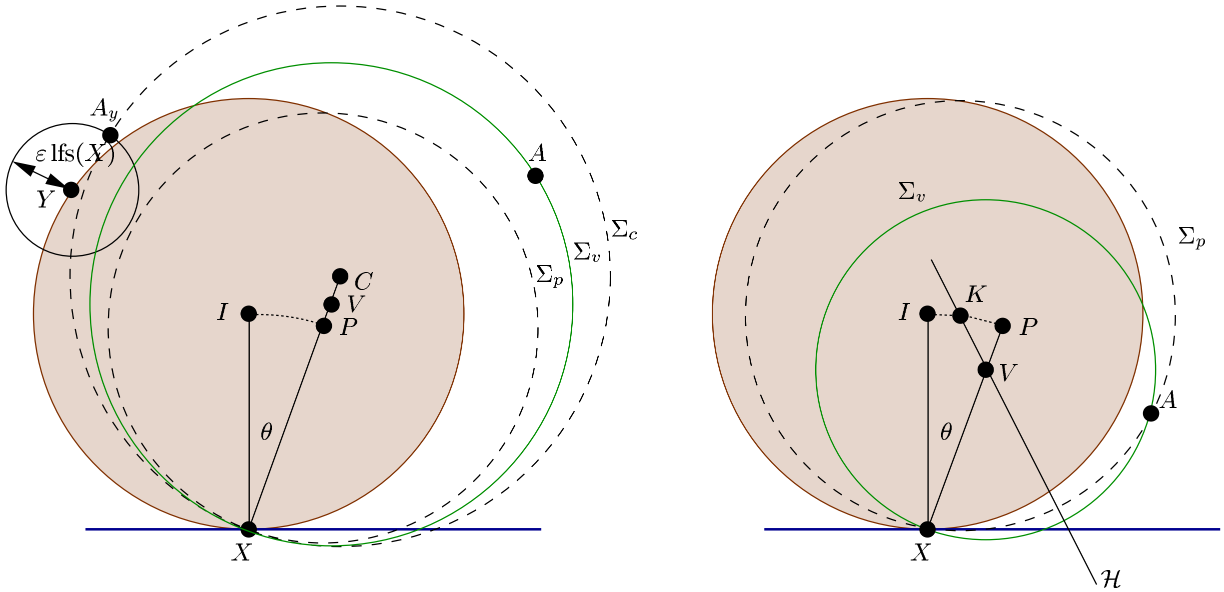


Figure 3: The two cases for the proof of Theorem 10

Proof. Since $||VX|| \geq \frac{1}{2}||XA||$ and $R \geq \mathrm{lfs}(X)$ and using Assumptions 1 and 3, we have:

$$\frac{\varepsilon}{||VX||} \leq \frac{2\varepsilon}{||XA||} \ll \frac{\varepsilon}{R}\sqrt{\frac{\mathrm{lfs}(X)}{\varepsilon}} \leq \sqrt{\frac{\varepsilon}{\mathrm{lfs}(X)}} \ll 1 \tag{1}$$

Therefore, we can apply Lemma 9. The angle θ between the vectors $\overrightarrow{XI}$ and $\overrightarrow{XV}$ is at most :

$$\theta \leq \arcsin\left(\frac{\varepsilon}{||VX||}\right) + \arcsin\left(\frac{\varepsilon}{\mathrm{lfs}(X)}\right) \approx \frac{\varepsilon}{||VX||} + \frac{\varepsilon}{\mathrm{lfs}(X)} \tag{2}$$

Let Σ_v and Σ_p be two spheres passing through X. The first one, Σ_v, is centered at V and is therefore empty. The second one, Σ_p, has radius R and is centered on the half-line going from X to V. We denote by P the center of Σ_p.

Two cases must be considered. First, assume V is farther from X than P (see Figure 3 left). In this case, the ball bounded by Σ_p is contained in the ball bounded by Σ_v and therefore does not contain any point of $\mathcal{A}$ in its interior. Let A_y be one of the sample points in the neighbourhood of Y at distance at most ε from Y, and Σ_c the sphere tangent to Σ_p at X and passing through A_y. We denote by C the center of Σ_c. Since V lies between C and P, $||VI|| \leq ||CI||$. We apply Lemma 8 with A_y, X, I and C. The five items of Lemma 8 are fulfilled. Indeed, since $||VX|| \geq ||PX|| = R \geq \mathrm{lfs}(X)$, Equation (2) implies $\theta \lesssim \frac{2\varepsilon}{\mathrm{lfs}(X)}$; as noticed above, $\rho \leq \frac{\varepsilon}{R} \leq \frac{\varepsilon}{\mathrm{lfs}(X)}$; and $2l = ||XA_y|| \geq ||XY|| - \varepsilon$. Let us first remark that $||XY|| \gg \varepsilon$. Indeed, using again the inequality $R \geq \mathrm{lfs}(X)$ and Assumptions 2 and 1, we have:

$$\frac{||XY||}{2} \gg R\sqrt{\frac{\varepsilon}{\mathrm{lfs}(X)}} \geq \sqrt{\mathrm{lfs}(X)\varepsilon} \gg \varepsilon \tag{3}$$

We now use our assumption $\frac{||XY||^2}{4R^2} \gg \frac{\varepsilon}{\mathrm{lfs}(X)}$ to prove that $l \gg R\theta$ and $l^2 \gg \rho R^2$. Indeed:

$$l \geq \frac{||XY||}{2} - \varepsilon \gg R\sqrt{\frac{\varepsilon}{\mathrm{lfs}(X)}} \gg R\frac{2\varepsilon}{\mathrm{lfs}(X)} \gtrsim R\theta$$

$$\text{and} \quad l^2 \geq \left(\frac{||XY||}{2} - \varepsilon\right)^2 \gg R^2\frac{\varepsilon}{\mathrm{lfs}(X)} \geq R^2\frac{\varepsilon}{R} \geq \rho R^2$$

Since $\frac{R}{||XY||} \geq \frac{1}{2}$, Lemma 8 then implies:

$$||VI|| \leq ||CI|| \lesssim \frac{R^2\theta}{l} + \frac{R^3\rho}{2l^2} \lesssim \frac{4R^2}{||XY||}\frac{\varepsilon}{\mathrm{lfs}(X)}\left(1 + \frac{R}{2||XY||}\right) \leq R\frac{\varepsilon}{\mathrm{lfs}(X)}\frac{10R^2}{||XY||^2} \tag{4}$$

Consider now the second case and assume that V is closer to X than P. Let $\mathcal{H}$ be the plane that bisects X and A. Since $\mathcal{H}$ contains V, P lies in the half-space limited by $\mathcal{H}$ that contains A. Since I lies in the other half-space, the circle arc IP (centered at X and of radius R) must intersect $\mathcal{H}$ at some point K. We now apply Lemma 8 to A, X, K and V. Notice that $\rho = 0$ and $\theta' = \angle(\overrightarrow{XK}, \overrightarrow{XV})$. Let $l = \frac{1}{2}||XA||$. Since $\rho = 0$, $l^2 \gg \rho R^2$. By Inequality 1, $\frac{\varepsilon}{||VX||} \ll \sqrt{\frac{\varepsilon}{\mathrm{lfs}(X)}}$ and thus:

$$\theta' \leq \theta \lesssim \frac{\varepsilon}{||VX||} + \frac{\varepsilon}{\mathrm{lfs}(X)} \ll \sqrt{\frac{\varepsilon}{\mathrm{lfs}(X)}} + \frac{\varepsilon}{\mathrm{lfs}(X)} \approx \sqrt{\frac{\varepsilon}{\mathrm{lfs}(X)}}$$

We now use our assumption $\frac{||XA||^2}{4R^2} \gg \frac{\varepsilon}{\mathrm{lfs}(X)}$ to prove that $l \gg R\theta'$:

$$l = \frac{1}{2}||XA|| \gg R\sqrt{\frac{\varepsilon}{\mathrm{lfs}(X)}} \gg R\theta'$$

Lemma 8 then gives:

$$||VK|| \lesssim \frac{R^2\theta'}{l}$$

Therefore:

$$\begin{aligned} ||VI|| &\leq ||VK|| + ||KI|| \\ &\lesssim \frac{R^2\theta}{l} + R\theta \\ &\lesssim R\varepsilon \left(\frac{1}{||VX||} + \frac{1}{\mathrm{lfs}(X)}\right)\left(1 + \frac{2R}{||XA||}\right) \\ &\lesssim R\frac{\varepsilon}{\mathrm{lfs}(X)}\left(1 + \frac{2R}{||XA||}\right)^2 \end{aligned}$$

□

Remark that the proposition above makes no assumption on the contact point Y of the maximal ball through X. Y may or may not be regular. It does not matter either that the local feature size vanishes at Y.

We can also remark that the result depends on the ratio $\frac{\varepsilon}{\mathrm{lfs}(X)}$. In fact, if $r = \frac{\varepsilon}{\mathrm{lfs}(X)}$, we get an equivalent approximation theorem for r-samples, as defined by Amenta and Bern in [2].

3 Polyhedral surfaces

3.1 Definition and properties

We call *polyhedral surface* the boundary of a bounded polyhedron with a finite number of faces. The medial axis of a polyhedral surface is composed of pieces of planes and quadrics (see Figure 6b). In this section, we establish four properties concerning polyhedral surfaces that will be used in the next section in order to bound the complexity of the Delaunay triangulation of points distributed on a polyhedral surface.

Property 11 *Let $\mathcal{S}$ be a polyhedral surface. For every facet $\mathcal{F} \subset \mathcal{S}$ and every point $X \in \mathcal{F}$, $B(X, \mathrm{lfs}(X)) \cap \mathcal{S}$ is a disk contained in $\mathcal{F}$.*

Proof. Refer to Figure 4. Let $\mathcal{S}_f = \mathcal{S} \setminus \mathcal{F}$ be the facets of $\mathcal{S}$ different from $\mathcal{F}$. To get a contradiction, we assume that $B(X, \mathrm{lfs}(X)) \cap \mathcal{S}_f \neq B(X, \mathrm{lfs}(X)) \cap \mathcal{F}$. We consider the point Y of $\mathcal{S}_f$ closest to X. Y belongs to a facet $\mathcal{F}' \neq \mathcal{F}$.

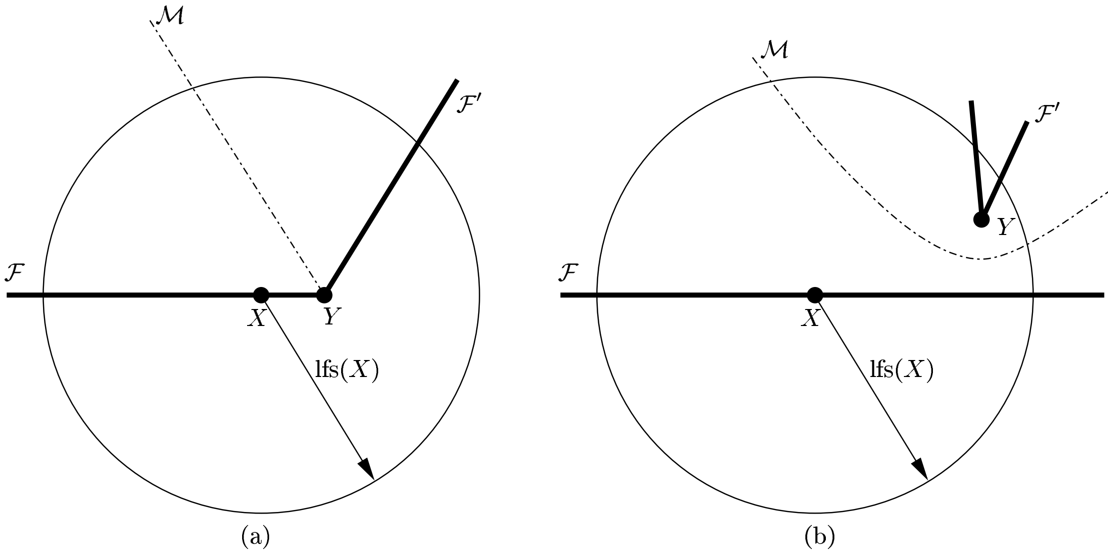


Figure 4: For the proof of Property 11.

If Y is on an edge of $\mathcal{F}$ (see Figure 4a), $||XY|| = \text{lfs}(X)$ since the edges of $\mathcal{S}$ are included in the medial axis of $\mathcal{S}$. Hence the intersection of $B(X, \text{lfs}(X))$ with the support plane of $\mathcal{F}$ is a disk $\mathcal{D} \subset \mathcal{F}$. Moreover, since $B(X, \text{lfs}(X)) \cap \mathcal{S}$ is a topological disk [4, Proposition 14], $B(X, \text{lfs}(X)) \cap \mathcal{S} = \mathcal{D}$.

If Y does not belong to $\mathcal{F}$ (see Figure 4b). In that case, $B(X, \text{lfs}(X)) \cap \mathcal{S}$ consists of at least two connected components, one included in $\mathcal{F}$ and the other containing Y. This contradicts the fact that $B(X, \text{lfs}(X)) \cap \mathcal{S}$ is a topological disk.

□

Property 12 *Let $\mathcal{S}$ be a polyhedral surface and X be a regular point of $\mathcal{S}$. Let B a maximal ball centered on the medial axis of $\mathcal{S}$ that passes through X and Y. Then* $\text{lfs}(X) \leq ||XY||$.

Proof. The proof is by contradiction. Let us assume $||XY|| < \text{lfs}(X)$. Since X is a regular point, X and Y do not belong to the same facet. Let $\mathcal{F}$ be the facet containing X. Y belongs to $B(X, \text{lfs}(X)) \cap \mathcal{S}$ and Y does not belong to $\mathcal{F}$, which contradicts Proposition 11. □

Property 13 *Let $\mathcal{P}$ be a convex polyhedron and $\mathcal{S}$ be the boundary of $\mathcal{P}$. Let R be the radius of the maximal ball (included in $\mathcal{P}$) that passes through $X \in \mathcal{S}$. For every regular point $X \in \mathcal{S}$, we have*

$$1 \leq \frac{R}{\text{lfs}(X)} \leq \frac{1}{\cos\frac{\alpha_{\max}}{2}},$$

where $\alpha_{\max} \in [0, \pi[$ designates the greatest angle between any two faces of $\mathcal{P}$.

Proof. Refer to Figure 5. We have already noticed that $1 \leq \frac{\text{lfs}(X)}{R}$. Let $\mathcal{F}$ be the facet of $\mathcal{S}$ containing X. Let $\text{Terr}(\mathcal{F}) = \{Y \in \mathcal{P}, d(Y, \mathcal{F}) = d(Y, \partial\mathcal{P}\}$. Since $\mathcal{P}$ is a convex polyhedron, $\text{Terr}(\mathcal{F})$ is a convex polyhedron bounded by $\mathcal{F}$ and the medial axis $\mathcal{M}$ of $\mathcal{P}$. Let J be a point of the medial axis closest to X. Observe that J cannot be on an edge of the medial axis since $\text{Terr}(\mathcal{F})$ is convex. Let $\mathcal{H}$ be the support plane of the facet of the medial axis containing J. The maximal ball centered at J touches $\mathcal{S}$ in two facets $\mathcal{F}$ and $\mathcal{F}'$. We denote by $\alpha \in [0, \pi[$ the angle between $\mathcal{F}$ and $\mathcal{F}'$. In addition, we call I the center of the maximal ball passing through X and K the intersection point of the plane $\mathcal{H}$ with the straight-line (XI). Since $\text{Terr}(\mathcal{F})$ is convex, $||XI|| \leq ||XK||$ and

$$\frac{R}{\text{lfs}(X)} = \frac{||XI||}{||XJ||} \leq \frac{||XK||}{||XJ||} = \frac{1}{\cos\frac{\alpha}{2}} \leq \frac{1}{\cos\frac{\alpha_{\max}}{2}}$$

□

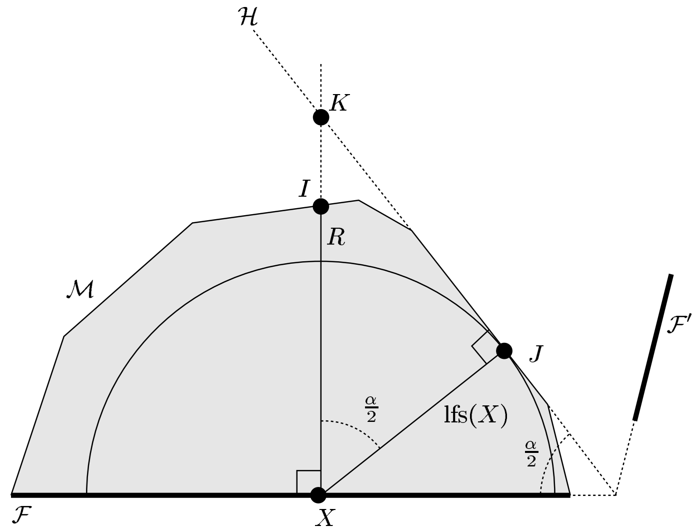


Figure 5: For the proof of Property 13. The gray region represents the set of points nearest from $\mathcal{F}$ than to any other facet.

3.2 Counting Delaunay edges

In the sequel, $\mathcal{S}$ designates a polyhedral surface and $\mathcal{A}$ a light uniform ε-sample of $\mathcal{S}$. We count the Delaunay edges incident to $X \in \mathcal{A}$. We enclose $\mathcal{S}$ in a sufficiently large bounding box $\mathcal{B}$. Therefore, the radius of any maximal ball remains bounded. We denote by $R_{\max}$ the radius of the greatest maximal ball. Moreover we add points on $\mathcal{B}$ so that the union of these additional points and the sample points on $\mathcal{S}$ constitute a light uniform ε-sample of $\mathcal{B} \cup \mathcal{S}$. Observe that the total number of points remains $O(n)$.

We consider two different types of zones on the surface (see Figure 6a), a l-singular zone surrounding singular points and a l-regular zone containing exclusively regular points.

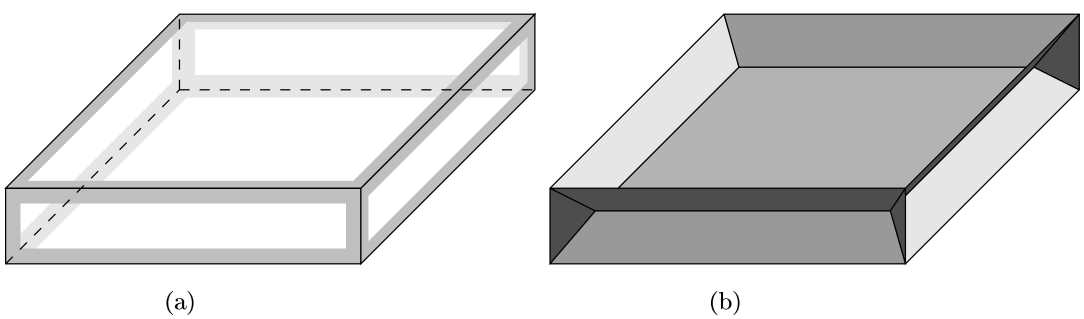


Figure 6: A polyhedral surface and its l-singular zone on the left. Its medial axis on the right.

Definition 14 (l-Regular and l-singular zones) *Let $l \geq 0$. We call* l-regular zone *of $\mathcal{S}$, the set of points $X \in \mathcal{S}$ such that* $\mathrm{lfs}(X) > l$*. We call* l-singular zone *of $\mathcal{S}$ the set of points that do not belong to the l-regular zone.*

The 0-singular zone (i.e. the set of singular points) of a polyhedral surface consists of its edges. For $l \neq 0$, the l-singular zone surrounds the edges of the polyhedral surface. The width of the l-singular zone around an edge depends on the dihedral angle at this edge. The parameter l will be fixed later on.

In the l-regular zone, every point X has only two types of Delaunay neighbours: neighbours that are "close" to X and neighbours that are "close" to the two maximal balls through X:

Proposition 15 *Let $l^3 \gg 4R_{\max}^2\varepsilon$. Let X be a sample point in the l-regular zone of $\mathcal{S}$ and A, a Delaunay neighbour of X. Let $B(I_0, R_0)$ and $B(I_1, R_1)$ be the two maximal balls through X. Then there exists*

$h_i \approx 20\, R_i^3\, \frac{\varepsilon}{l^3}$ *for* $i \in \{0,1\}$ *such that:*

$$A \in B(X, \mathrm{lfs}(X)) \cup B\left(I_0, R_0 + h_0\right) \cup B\left(I_1, R_1 + h_1\right)$$

Proof. Let V be any vertex of the Voronoi facet dual to the Delaunay edge $[XA]$. Let I and R be the center and the radius of the maximal ball through X such that $\overrightarrow{XV} \cdot \overrightarrow{XI} > 0$. We assume $||XA|| \geq \mathrm{lfs}(X)$ and prove that there exists $h \approx 20\, R^3\, \frac{\varepsilon}{l^3}$ such that $A \in B\left(I, R + h\right)$.

In order to apply Theorem 10, we have to check that $\frac{||XY||^2}{4R^2} \gg \frac{\varepsilon}{\mathrm{lfs}(X)}$ and $\frac{||XA||^2}{4R^2} \gg \frac{\varepsilon}{\mathrm{lfs}(X)}$. But, by Proposition 12, $||XY|| \geq \mathrm{lfs}(X)$ and:

$$\frac{||XY||^2}{4R^2} \geq \frac{\mathrm{lfs}(X)^2}{4R^2} \geq \frac{l^2}{4R^2_{\mathrm{max}}} \gg \frac{\varepsilon}{l} \geq \frac{\varepsilon}{\mathrm{lfs}(X)}$$

In a similar way:

$$\frac{||XA||^2}{4R^2} \geq \frac{\mathrm{lfs}(X)^2}{4R^2} \geq \frac{l^2}{4R^2_{\mathrm{max}}} \gg \frac{\varepsilon}{l} \geq \frac{\varepsilon}{\mathrm{lfs}(X)}$$

Therefore and using the inequality $1 \leq \frac{R}{\mathrm{lfs}(X)}$:

$$||VI|| \lesssim R\, \frac{\varepsilon}{\mathrm{lfs}(X)}\, \max\left(\frac{10R^2}{\mathrm{lfs}(X)^2}, \left(1 + \frac{2R}{\mathrm{lfs}(X)}\right)^2\right) \lesssim 10\, R^3\, \frac{\varepsilon}{l^3}$$

An upper bound on $||IA||$ follows immediately since, by triangle inequality:

$$\begin{aligned} ||IA|| &\leq ||IV|| + ||VA|| = ||IV|| + ||VX|| \\ &\leq 2||IV|| + ||IX|| \\ ||IA|| - R &\lesssim 20\, R^3\, \frac{\varepsilon}{l^3} \end{aligned}$$

□

Lemma 16 *Let* $l > 2\varepsilon$. *Let* X *be a point in the* l*-regular zone. The number of Delaunay neighbours of* X *in* $B(X, \mathrm{lfs}(X))$ *is* $O(1)$.

Proof. By Property 11, we know that $B(X, \mathrm{lfs}(X)) \cap \mathcal{S}$ is a disk $\mathcal{D}$. Let $A \in \mathcal{D}$ be a Delaunay neighbour of X. Let $\mathcal{B}$ be an empty ball passing through X and A, i.e. a ball whose interior contains no sample points. $\mathcal{B}$ intersects the support plane of $\mathcal{D}$ in a disk centered at C. Let I be the point of the half-line starting at X and passing through C such that $||XI|| = \min\left(||XC||, \frac{\mathrm{lfs}(X)}{2}\right)$. By definition of I, $B(I, ||XI||) \subseteq B(X, \mathrm{lfs}(X))$. Because $B(X, \mathrm{lfs}(X))$ intersects the surface in a disk, $B(I, ||XI||)$ also intersects the surface in a disk, and, since $B(I, ||XI||) \subset \mathcal{B}$, $B(I, ||XI||)$ is empty. Our sampling condition imposes that $||XI|| = \min\left(||XC||, \frac{\mathrm{lfs}(X)}{2}\right) \leq \varepsilon$, which implies that $||XC|| \leq \varepsilon$ since $\mathrm{lfs}(X) > 2\varepsilon$. Therefore, $||XA|| \leq 2||XC|| \leq 2\varepsilon$. Since the sample is light, the number of sample points at distance 2ε from X is $O(1)$. □

Proposition 17 (Counting edges in the l-regular zone) *Let* $l^3 \gg 4R^2_{\mathrm{max}}\varepsilon$. *The number of Delaunay edges incident to a given sample point of the regular zone is* $O(\frac{1}{\varepsilon\, l^3})$. *The total number of edges incident to the* l*-regular zone is* $O(\frac{1}{\varepsilon^3\, l^3})$.

Proof. Let X be any point in the l-regular zone. Let $B(I_0, R_0)$ and $B(I_1, R_1)$ be the two maximal balls through X. By Proposition 15, there exists $h_i \approx 20\, R_i^3\, \frac{\varepsilon}{l^3}$ for $i \in \{0,1\}$ such that the Delaunay neighbours of X belong to:

$$B(X, \mathrm{lfs}(X)) \cup B\left(I_0, R_0 + h_0\right) \cup B\left(I_1, R_1 + h_1\right).$$

By Lemma 16, the number of Delaunay neighbours in $B(X, \mathrm{lfs}(X))$ is $O(1)$. Let us prove that the number of Delaunay neighbours in $B\left(I_i, R_i + h_i\right)$ is $O(\frac{1}{\varepsilon\, l^3})$, for $i \in \{0,1\}$. Because the polyhedral surface has a

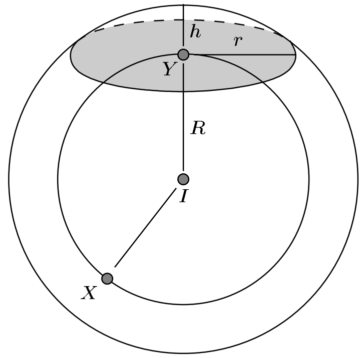


Figure 7: A plane tangent to the ball $B(I, R)$ at Y intersects $B(I, R+h)$ in a disk of radius $r = \sqrt{2hR - h^2} \approx \sqrt{2Rh}$ when $h \ll R$.

bounded number of faces, the intersection of the enlarged maximal ball $B(I_i, R_i + h_i)$ with $\mathcal{S}$ is included in a bounded number of disks. Each disk has radius at most $r \approx \sqrt{2R_i h_i} \approx R_i^2 \sqrt{\frac{40\varepsilon}{l^3}}$ (see Figure 7). Therefore, the number of sample points in the enlarged maximal ball is $O(\frac{r^2}{\varepsilon^2}) = O(\frac{1}{\varepsilon\, l^3})$. $\square$

Proposition 18 (Counting edges in the l-singular zone) *Let $l > 0$. The number of Delaunay edges joining two sample points in the l-singular zone is $O(\frac{l^2}{\varepsilon^4})$.*

Proof. Let p be the length of the 0-singular zone. The l-singular zone can be covered by $O(\frac{p}{l})$ spheres of radius $2l$. Therefore, the number of points in the l-singular zone is $O(\frac{p}{l} \times \frac{4l^2}{\varepsilon^2}) = O(\frac{l}{\varepsilon^2})$. The number of edges joining two points of the l-singular zone is therefore $O(\frac{l^2}{\varepsilon^4})$. $\square$

In order to counterbalance the number of edges in the two zones, we have to choose l such that $\frac{l^2}{\varepsilon^4} = \frac{1}{\varepsilon^3 l^3}$, in other words $l = \sqrt[5]{\varepsilon}$. We sum up our results in the following theorem :

Theorem 19 *Let $\mathcal{A}$ be a light uniform ε-sample of a bounded polyhedral surface $\mathcal{S}$ of $\mathbb{R}^3$. The number of tetrahedra of the Delaunay triangulation of $\mathcal{A}$ is $O(n^{\frac{9}{5}}) = O(n^{1.8})$.*

4 Convex polyhedral surfaces

For convex polyhedral surfaces, we know from Proposition 13 that the ratio $\frac{R}{\mathrm{lfs}(X)}$ is bounded from above by a certain constant $C = \left(\cos \frac{\alpha_{\max}}{2}\right)^{-1}$ where $\alpha_{\max} < \pi$ represents the greatest angle between any two facets. Using the same scheme as before, one can prove that the complexity of convex polyhedral surfaces is $O(n\sqrt{n})$. Again, we consider two different zones on the convex polyhedral surface.

Proposition 20 *Let $l \gg 4C^2\varepsilon$. Let X be a sample point in the l-regular zone of $\mathcal{S}$ and A, a Delaunay neighbour of X. Let $B(I_0, R_0)$ and $B(I_1, R_1)$ be the two maximal balls through X. Then, there exists $h \approx 20C^2\varepsilon$ such that:*

$$A \in B(X, \mathrm{lfs}(X)) \cup B(I_0, R_0 + h) \cup B(I_1, R_1 + h)$$

Proof. Let V be any vertex of the Voronoi facet dual to the Delaunay edge $[XA]$. Let I and R be the center and the radius of the maximal ball through X such that $\overrightarrow{XV} \cdot \overrightarrow{XI} > 0$ and assume $\|XA\| \geq \mathrm{lfs}(X)$. Let us prove that there exists $h \approx 20C^2\varepsilon$ such that $A \in B(I, R+h)$.

Using $\frac{1}{4C^2} \gg \frac{\varepsilon}{l}$ and Property 12, we get :

$$\frac{\|XY\|^2}{4R^2} \geq \frac{\mathrm{lfs}(X)^2}{4R^2} \geq \frac{1}{4C^2} \gg \frac{\varepsilon}{l} \geq \frac{\varepsilon}{\mathrm{lfs}(X)}$$

In a similar way:

$$\frac{||XA||^2}{4R^2} \geq \frac{\text{lfs}(X)^2}{4R^2} \geq \frac{1}{4C^2} \gg \frac{\varepsilon}{l} \geq \frac{\varepsilon}{\text{lfs}(X)}$$

Therefore, by Theorem 10 and using $1 \leq C$:

$$\begin{aligned} ||VI|| &\lesssim R \frac{\varepsilon}{\text{lfs}(X)} \max\left(\frac{10R^2}{\text{lfs}(X)^2}, \left(1 + \frac{2R}{\text{lfs}(X)}\right)^2\right) \\ &\lesssim \max\left(10C^2, (1+2C)^2\right)\varepsilon \\ &\lesssim 10C^2\varepsilon \end{aligned}$$

and

$$||IA|| - R \lesssim 20C^2\,\varepsilon$$

□

Proposition 21 (Counting edges in the l-regular zone) *Let $l \gg 4C^2\varepsilon$. Let $\mathcal{A}$ be a light uniform ε-sample. The number of Delaunay edges incident to a given sample point of the l-regular zone is $O(\sqrt{n})$. The total number of edges incident to the l-regular zone is $O(n\sqrt{n})$.*

Proof. By Lemma 16, the number of Delaunay neighbours in $B(X, \text{lfs}(X))$ is $O(1)$. Let us prove that the number of points in the enlarged maximal ball $B(I, R+h)$ is $O(\sqrt{n})$ where $h \approx 20C^2\varepsilon$. The intersection of the enlarged maximal ball with $\mathcal{S}$ is included in a bounded number of disks. Each disk has radius at most $r \approx \sqrt{2Rh}$ (see Figure 7). Therefore, the number of sample points in the enlarged maximal ball is $O(\frac{r^2}{\varepsilon^2}) = O(\frac{1}{\varepsilon}) = O(\sqrt{n})$. □

The counting of edges in the l-singular zone is unchanged and is given by Proposition 18. In order to find $O(n\sqrt{n}) = O(\frac{1}{\varepsilon^3})$ edges in the singular zone, we have to choose $\frac{l^2}{\varepsilon^4} = \frac{1}{\varepsilon^3}$, in other words $l = \sqrt{\varepsilon}$.

Theorem 22 *Let $\mathcal{A}$ be a light uniform ε-sample of a bounded convex polyhedral surface $\mathcal{S}$ of $\mathbb{R}^3$. The number of tetrahedra of the Delaunay triangulation of $\mathcal{A}$ is $O(n\sqrt{n})$.*

5 Conclusion

We have given a sub-quadratic bound on the complexity of the Delaunay triangulation of points distributed on polyhedral surfaces. Our proof is based on the fact that a subset of the Voronoi vertices of a sample are close to the medial axis of the surface. This result of independent interest holds also for smooth surfaces. An obvious open question is to extend our results to smooth surfaces. Bounding the number of tetrahedra in the case of smooth surfaces seems to be harder, due to the presence of slivers, i.e. flat tetrahedra whose circumcenters can be arbitrarily far from the medial axis. Observe that Theorem 10 indicates that these tetrahedra are small.

Note: Since the publication of this paper, we established a linear bound for polyhedral surfaces using a different approach[1]. Golin and Na established also a new result in the probalistic context[2].

References

[1] N. Amenta and R. K. Kolluri. The medial axis of a union of balls. *Comput. Geom. Theory Appl.*, 20:25–37, 2001.

[1]Dominique Attali and Jean-Daniel Boissonnat. A linear bound on the complexity of the Delaunay triangulation of points on polyhedral surfaces. In *Proc. 7th ACM Symposium on Solid Modeling and Applications*, pages 139–145, 2002.

[2]Mordecai J. Golin and Hyeon-Suk Na. The probabilistic complexity of the voronoi diagram of points on a polyhedron. In *Proc. 14th Annu. ACM Sympos. Comput. Geom.*, pages 209–216, 2002.

[2] Nina Amenta and Marshall Bern. Surface reconstruction by voronoi filtering. *Discrete Comput. Geom.*, 22:481–504, 1999.

[3] A. G. Belyaev, E. V. Anoshkina, and T. L. Kunii. Ridges, ravines and singularities. In A.T. Fomenko and T. L. Kunii, editors, *Topological Modeling for Visualization*. Springer, 1997.

[4] J.-D. Boissonnat and F. Cazals. Natural neighbor coordinates of points on a surface. *Comput. Geom. Theory Appl.*, 19:155–173, 2001.

[5] Jean-Daniel Boissonnat and Frédéric Cazals. Smooth surface reconstruction via natural neighbour interpolati on of distance functions. *Comp. Geometry Theory and Applications*, pages 185–203, 2002.

[6] T. M. Chan, J. Snoeyink, and C. K. Yap. Primal dividing and dual pruning: Output-sensitive construction of 4-d polytopes and 3-d Voronoi diagrams. *Discrete Comput. Geom.*, 18:433–454, 1997.

[7] Hyeong In Choi, Sung Woo Choi, and Hwan Pyo Moon. Mathematical theory of medial axis transform. *Pacific Journal of Mathematics*, 181(1):57–88, 1997.

[8] R. A. Dwyer. Higher-dimensional voronoi diagram in linear expected time. *Discrete Comput. Geom.*, 6:343–367, 1991.

[9] R. A. Dwyer. The expected number of k-faces of a voronoi diagram. *Internat. J. Comput. Math.*, 26(5):13–21, 1993.

[10] Jeff Erickson. Nice point sets can have nasty delaunay triangulations. In *17th Annual ACM Symposium on Computational Geometry*, pages 96–105, 2001.

[11] Mordecai Golin and HyeonSuk Na. On the average complexity of 3d-voronoi diagrams of random points on convex polytopes. In *Proc. 12th Canad. Conf. Comput. Geom.*, pages 127–135, 2000.

[12] Riley J. and Calabi L. Certain properties of circles inscribed in simple closed curves. Technical Report 59281, Parke Math. Lab. Inc., One River Road, Carlisle, Massachusetts, 1964.

[13] Calabi L. and Riley J.A. The skeletons of stable planesets. Technical report, Parke Math. Lab. Inc., One River Road, Carlisle, Massachusetts, December 1967.

[14] G. Matheron. On the negligibility of the skeleton and the absolute continuity of erosions. In J. Serra, editor, *Image Analysis and Mathematical Morphology*, volume 2. Academic Press, London, 1988.